OPTIMAL ROOT RECOVERY FOR UNIFORM ATTACHMENT TREES AND d-REGULAR GROWING TREES

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Abstract. We consider root-finding algorithms for random rooted trees grown by uniform attachment. Given an unlabeled copy of the tree and a target accuracy $\varepsilon > 0$, such an algorithm outputs a set of nodes that contains the root with probability at least $1-\varepsilon$. We prove that, for the optimal algorithm, an output set of size $\exp(O(\sqrt{\log(1/\varepsilon)}))$ suffices; this bound is sharp and answers a question from [7]. We prove similar bounds for random regular trees that grow by uniform attachment, strengthening a result from [15].

1. Introduction

A sequence $(T_n)_{n\geq 1}$ of trees is said to follow the *uniform attachment* or *UA* model if it is generated as follows: T_1 consists of a single node (the root, denoted by $\varnothing$); for each $n \geq 1$, the tree T_{n+1} is generated from T_n by attaching a new node to an existing node chosen uniformly at random, independently of the previous choices. If $(T_n)_{n\geq 1}$ follows the UA model then write $T_n \sim \mathrm{UA}(n)$.

Root finding asks the following question: on observing T_n but not the identity of the root $\varnothing$, with what confidence can $\varnothing$ be recovered? More precisely, *a root-finding algorithm* is a function A that, given an (unlabeled, finite) input tree T, outputs a set $A(T)$ of nodes of T. The *size* of A is the function $s(A) := \sup(|A(T)|, T \text{ a finite tree})$. The *error* of A is $\sup(\mathbb{P}(\varnothing \notin A(T_n)), n \geq 1)$, where $T_n \sim \mathrm{UA}(n)$. In other words, it is the worst-case probability that the algorithm fails to return the root when the input is a uniform attachment tree of some size. Bubeck, Devroye and Lugosi [7] showed that for all $\varepsilon > 0$, there exists a UA root-finding algorithm A^ε with error at most ε and size at most $\exp(c\log(1/\varepsilon)/\log\log(1/\varepsilon))$, where $c > 0$ is a universal constant. They also showed that any root finding algorithm A with error at most ε must have size at least $\exp(c'\sqrt{\log 1/\varepsilon})$, for another universal constant $c' > 0$, and posed the question of whether either of these bounds are tight as an open problem.

The first main result of this work is to show that the above lower bound is tight, up to the value of c'. In order to precisely state this result, we first describe the algorithm we study. Fix a finite tree $T = (V, E)$ with $|T| := |V| < \infty$. Define a *centrality measure* $\varphi_T : V \to \mathbb{R}$ by setting

$$\varphi_T(u) = \prod_{v\in T\setminus\{u\}} |(T,u)_{v\downarrow}| \tag{1.1}$$

for $u \in V$; here (T,u) denotes the tree T rooted at node u and $(T,u)_{v\downarrow}$ denotes the subtree rooted at v in (T,u). Next, write $n = |T|$ and order the nodes of T as $(v_1,\ldots,v_n) = (v_1(T),\ldots,v_n(T))$ so that $\varphi_T(v_1) \leq \varphi_T(v_2) \leq \ldots \leq \varphi_T(v_n)$, breaking ties arbitrarily. Given a positive integer k, for an input tree T with $|T| = n$, the algorithm $\mathcal{A}_k$ has output $\mathcal{A}_k(T) = \{v_1(T),\ldots,v_{k\wedge n}(T)\}$ consisting of the nodes with the k smallest φ_T values (or

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all nodes, if the input T has fewer than k nodes). Clearly, $\mathcal{A}_k$ has size k. This algorithm turns out to have minimal error among algorithms of size k, for a fairly wide range of growing tree models, including the ones studied in this paper [10, Theorem 3].
In the next theorem, and throughout, write $\mathbb{N}_1$ for the set of strictly positive integers.

Theorem 1.1. *There exist $C^*, c^* > 0$ such that the following holds. For $\varepsilon > 0$, let $K = K(\varepsilon) = C^* \exp(c^* \sqrt{\log 1/\varepsilon})$. Then for all $n \in \mathbb{N}_1$, for $T_n \sim \mathrm{UA}(n)$, it holds that $\mathbb{P}(\varnothing \notin \mathcal{A}_K(T_n)) \leq \varepsilon$.*

The second main result of this work establishes an analogous bound for a degree-bounded variant of the UA model, previously studied in [15]. Given a positive integer $d \geq 3$, a *d-regular tree* is a tree in which every non-leaf node has exactly d neighbours. A sequence $(T_n)_{n\geq 1}$ of trees follows the *d-regular uniform attachment* or UA_d model if it is generated as follows. First, T_1 consists of a single root node $\varnothing$ and d leaves. Then, for each $n \geq 1$, T_{n+1} is built from T_n by adding $d-1$ new neighbours to a single leaf V_n of T_n, with V_n chosen uniformly at random from among the leaves of T_n, independently of all previous choices. Note that for all n, T_n is a d-regular tree with exactly n non-leaf nodes. If $(T_n)_{n\geq 1}$ follows the UA_d model then we write $T_n \sim \mathrm{UA}_d(n)$.

Theorem 1.2. *For any $d \geq 3$, there exist $C_d^*, c_d^* > 0$ such that the following holds. For $\varepsilon > 0$, let $K = K(\varepsilon) = C_d^* \exp(c_d^* \sqrt{\log 1/\varepsilon})$. Then for all $n \in \mathbb{N}_1$, for $T_n \sim \mathrm{UA}_d(n)$, it holds that $\mathbb{P}(\varnothing \notin \mathcal{A}_K(T_n)) \leq \varepsilon$.*

The previous theorem strengthens [15, Corollary 1], which showed that there exists an algorithm $\mathcal{A}'$ that returns a set of $O(1/\varepsilon)$ nodes satisfying that $\mathbb{P}(\varnothing \notin \mathcal{A}'(T_n)) \leq \varepsilon$. (The algorithm $\mathcal{A}'$ considered in (cite) ranks the nodes in increasing order of $\varphi'_T(u) := \max_v |(T,u)_{v\downarrow}|$, the maximum taken over all neighbours v of u in T, then returns the nodes with the smallest φ'_T values.) The bound in Theorem 1.2 is optimal up to the value of c_d^*; the proof of this consists in adapting a construction from [7], in which T_n grows by first building a long path stretching away from the root and then growing exclusively in the subtree at the far end of the path, to the d-regular setting. The details of this construction are tedious, but require no new ideas, so we omit them.

The mathematical study of root finding was instigated in [19], which showed that the optimal size-1 algorithm for the UA_3 model has error $1/4$, and that for each $d \geq 3$ there exists $\alpha_d \in (0, 1/2)$ such that the optimal size-1 algorithm for the UA_3 model has error α_d. (Note that a size-1 algorithm is simply allowed to output a single node as its guess for the identity of the root.) The paper [7] was the first to investigate the dependence between the error and the size; that work studied both uniform attachment trees and *preferential attachment* trees, in which the probability V_n connects to a node w is proportional to the current degree of w. Since then, root reconstruction has been studied for a wide range of growing tree [3, 6, 11, 17, 18] and graph [4, 5, 13] models. However, until quite recently, the best possible performance of a root finding algorithm (in terms of how the output set size depends on the error tolerance ε) was not known for any model. The only other such tight bound we are aware of is for the preferential attachment model, for which [9] proved that a lower bound established in [7] is in fact sharp.

1.1. A sketch of the proof

We begin by sketching the proof for the UA_d model (Theorem 1.2), as it is more straightforward; we then briefly discuss the adaptations needed to handle the uniform attachment model.
The proof of Theorem 1.2 consists of three key steps.

(I) Let $D-1$ be the greatest distance from the root of T_n of a node u with the property that $|(T_n, \varnothing)_{u\downarrow}|/|T_n| \geq 1/3$. Then D has exponential tails: there exists $C > 0$ such that $\mathbb{P}(D \geq m) \leq C(2/3)^m$. (See Lemma 3.3, below.)

This first point essentially follows by a union bound. (The constant C we obtain depends on d in the UA_d model, but we believe this dependence can be straightforwardly eliminated.)

(II) For a tree $T = (V, E)$ with root $\varnothing$, let $\Phi(T) = \varphi_T(\varnothing)/\min_{u\in V}\varphi_T(u)$. We call this the *competitive ratio* of T; it is the ratio of the centrality of $\varnothing$ to that of the most central node of T. Then $\Phi(T_n)$ has polynomial tails: $\mathbb{P}(\Phi(T_n) \geq x) \leq Ax^{-a}$ for some universal constants $A, a > 0$, not depending on d. (See Proposition 3.6, below.)

To prove this, we characterize the path $\varnothing = u_0, \ldots, u_G = v$ from the root to the (with high probability) unique node v with $\varphi_{T_n}(v)$ minimal, then bound G, and ultimately the competitive ratio $\varphi_{T_n}(\varnothing)/\varphi_{T_n}(v)$, by a model-dependent analysis of the splitting of mass in subtrees of T_n. This analysis uses and builds on the "nested Pólya urn" perspective introduced in [7]. The model-dependence arises here because different values of d yield different urn models; but the resulting constants a, A do not depend on d.

(III) For any integer $d \geq 3$, there exists $c = c(d) > 0$ such that for any tree $T = (V, E)$ with root $\varnothing$ where each node has at most $d-1$ children, if $u \in V$ is such that $|(T,\varnothing)_{u\downarrow}| \leq \frac{1}{3}|T|$ then $\left|\{v \in (T,\varnothing)_{u\downarrow} : \varphi_T(\varnothing) \geq \varphi_T(v)\}\right| \leq \exp(c + c\sqrt{\log\Phi(T)})$. This deterministically bounds the number of nodes which are more competitive candidates than the root which lie in a given small subtree of T. (See Proposition 3.5, below.)

The dependence of c on d in (III) seems unavoidable, as briefly discussed at the end of this section, and explained in greater detail later, in Section 3.4. However, in Section 4.5 we explain how to remove the d-dependence from the constant c_d^* in Theorem 1.2.

The idea behind the proof of (III) is that, once $|(T,\varnothing)_{u\downarrow}|$ is small, the function φ_T should increase quickly as one moves further into $|(T,\varnothing)_{u\downarrow}|$. Specifically, it is not hard to show that if $|(T,\varnothing)_{u\downarrow}| \leq |T|/3$ then for any child v of u, $\varphi_T(v) \geq \varphi_T(u)\cdot(|T|/|(T,\varnothing)_{u\downarrow}|-1) > 2|\varphi_T(u)|$. Combining this observation with deterministic arguments similar in spirit to those in [7], which involve some combinatorial reductions followed by the use of the Hardy-Ramanujan formula on the number of partitions of an integer, the bound in (III) follows.

We now use (I)-(III) to show Theorem 1.2 in full, then conclude the proof sketch by briefly discussing the adaptations needed to prove Theorem 1.1. (These adaptations are rather non-trivial, and somewhat technical, and we only outline them at a high level.)

Proof of Theorem 1.2. Write $B_n = \{v \in T_n : \varphi_{T_n}(\varnothing) \geq \varphi_{T_n}(v)\}$ for the set of nodes which are at least as central as the root, and let $H_n = \{u \in T_n : |(T_n,\varnothing)_{u\downarrow}| \geq |T_n|/3\}$. Note that H_n is a connected set of vertices which contains $\varnothing$, so in particular forms a subtree of T_n, and that by the pigeonhole principle H_n contains at most three leaves.
For $v \in T_n \setminus H_n$, there is a unique ancestor u of v which is a child of an element of H_n but which does not lie in H_n itself. Since each node of H_n has at most $d-1$ children in $T_n \setminus H_n$, we then have

$$|B_n| \leq |H_n| + (d-1)|H_n| \max_{u\in T_n\setminus H_n} |B_n \cap (T_n,\varnothing)_{u\downarrow}|.$$

Since H_n has at most 3 leaves, by the definition of D in (I) we have $|H_n| \leq 3D$. Next, for all $u \in T_n \setminus H_n$, by (III) we have $|B_n \cap (T_n,\varnothing)_{u\downarrow}| \leq \exp(c + c\sqrt{\log\Phi(T_n)})$, so the preceding displayed bound entails that

$$|B_n| \leq 3D\big(1 + (d-1)\exp(c + c\sqrt{\log\Phi(T_n)})\big) \leq 3dD\exp(c + c\sqrt{\log\Phi(T_n)})\,.$$

It follows that

$$\begin{aligned}\mathbb{P}\Big(|B_n| \geq 3d\log_{3/2}(\tfrac{1}{\varepsilon})\exp(c + c\sqrt{\tfrac{1}{a}\log\tfrac{1}{\varepsilon}})\Big) &\leq \mathbb{P}\big(D \geq \log_{3/2}\tfrac{1}{\varepsilon}\big) + \mathbb{P}\big(\Phi(T_n) \geq \tfrac{1}{\varepsilon^{1/a}}\big)\\ &\leq C\varepsilon + A\varepsilon\end{aligned}$$

by the bounds in (I) and (II). Since $\varnothing \in \mathcal{A}_K(T_n)$ whenever $K \geq |B_n|$, the result follows. □

The proof of Theorem 1.1 follows the same basic strategy as that of Theorem 1.2, but several adaptations are needed. First, to help address the fact that degrees are unbounded in the UA model, in step (I), rather than work with *distances* we work with *weights*. Weights are defined inductively; the root has weight $\mathrm{w}(\varnothing) = 0$, and if v is the k'th child of u then we set $\mathrm{w}(v) = \mathrm{w}(u) + k$. With this adjustment, versions of (I) and (II) for the UA model follow using similar (though more technical) arguments to those for the UA_d model. (See respectively Lemma 4.2 and Proposition 4.7, below.)
The most challenging adaptation is that there is no fixed constant $c > 0$ which makes (III) true deterministically for all trees. (To see this, consider the case where T is a star rooted at a leaf.) We are thus obliged to resort to probabilistic bounds; we replace the inequality in (III) by something which has the flavour of the following statement: for $T_n \sim \mathrm{UA}(n)$, the ratio $\left|\{v \in (T_n, \varnothing)_{u\downarrow} : \varphi_{T_n}(\varnothing) \geq \varphi_{T_n}(v)\}\right| / \exp(c + c\sqrt{\log \Phi(T_n)})$ has exponentially decaying upper tail, uniformly in n. This is not exactly what we prove, but the precise statement requires more technical setup than is suitable for a proof overview. (See Proposition 4.5.) Let us at least remark that the proof of the precise statement relies on an on-average geometric decay of the sizes of the subtrees stemming from the children of a given node; this is proved via another analysis of the model.

1.2. Overview of the paper

Section 2 contains material which is used in the proofs of both main results: a formalism of the model that we use for the analysis; some basic facts about the centrality measure; an important bound for the behaviour of certain geometrically decaying flows on trees (Proposition 2.4); and some definitions, distributional identities for, and relations between different families of random variables. Section 3 contains the proofs of the three key steps used to show Theorem 1.2 in Section 1.1, as well as a brief discussion of the d-dependence of the bounds of that theorem (see Section 3.4). Section 4 contains the proof of Theorem 1.1, followed by a sketch of how the ideas from its proof can be adapted to replace the d-dependent constant c^*_d from Theorem 1.2 by a universal constant c^*; see Section 4.5.

2. Setup and general results

This section contains formalism and results that are used in proving both Theorems 1.1 and 1.2. The first subsection, below, introduces the *Ulam–Harris* formalism for rooted ordered trees, which we use for all of the analysis. The second subsection establishes some deterministic facts connecting centrality measure and the competitive ratios of tree nodes. The third subsection controls the behaviour of certain geometrically decaying functions on rooted trees, that will be crucial for our analysis. The final subsection recalls some important distributions that naturally arise in the analysis and gives some of their basic properties.

2.1. The Ulam–Harris formalism

Recall that $\mathbb{N}_1 = \{1, 2, 3, \ldots\}$ is the set of positive integers and let $\mathbb{U}$ be the set of finite words written on the alphabet $\mathbb{N}_1$, namely

$$\mathbb{U} = \bigcup_{n \geq 0} \mathbb{N}_1^n, \qquad \text{with the convention that } \mathbb{N}_1^0 = \{\varnothing\}.$$

For two words $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$, let $u * v = (u_1, \ldots, u_n, v_1, \ldots, v_m)$ stand for the concatenation of u and v. If u is not the empty word $\varnothing$, then set $\overleftarrow{u} = (u_1, \ldots, u_{n-1})$. We further call $\overleftarrow{u}$ the *parent* of u and say that u is a *child* of $\overleftarrow{u}$. For any $j \geq 1$, we interpret $u * j := u * (j)$ as the j'th child of u. This yields a *genealogical order*

denoted by $\preceq$: for $u, v \in \mathbb{U}$, we write $u \preceq v$ if and only if there exists $w \in \mathbb{U}$ such that $v = u * w$ (in other words, u is a prefix of v). In that case, say that u is an *ancestor* of v and that v is a *descendant* of u. Note that u is always an ancestor and a descendant of itself. We also use the notation $u \prec v$ to express that $u \preceq v$ and $u \neq v$.
It is useful to quantify the "size" of any word $u = (u_1, \ldots, u_n) \in \mathbb{U}$ we encounter. First, set $\mathrm{ht}(u) = n$, so that $\mathrm{ht}(\varnothing) = 0$ and $\mathrm{ht}(u) = 1 + \mathrm{ht}(\overleftarrow{u})$ when $u \neq \varnothing$, and call $\mathrm{ht}(u)$ the or height of u. Second, define the *weight* of u as the sum of its letters: namely,

$$\mathrm{w}(u) = \sum_{i=1}^{n} u_i. \tag{2.1}$$

This exactly corresponds to the weight function defined in Section 1.1. For integer $d \geq 2$ write $\mathbb{U}_d$ for the subset of $\mathbb{U}$ consisting of words $u = (u_1, \ldots, u_n)$ with $u_1 \in [d+1]$ and $u_i \in [d]$ for $2 \leq i \leq n$. The set $\mathbb{U}_{d+1}$ will be useful when studying the UA_{d+1} model.

Definition 2.1. *A* plane tree *is a finite set $t \subset \mathbb{U}$ that satisfies the following properties:*

(a) *it holds that $\varnothing \in t$;*
(b) *for all $u \in t \setminus \{\varnothing\}$, it holds that $\overleftarrow{u} \in t$;*
(c) *for all $u \in t$, there exists an integer $\mathrm{k}_u(t) \geq 0$ such that for any $j \geq 1$, $u * j \in t \iff 1 \leq j \leq \mathrm{k}_u(t)$.*

We view $\varnothing$ as the root of any plane tree. For a plane tree t and for $u \in t$, we call the integer $\mathrm{k}_u(t)$ the *number of children of u in t*. When $\mathrm{k}_u(t) = 0$, we say that u is a *leaf* of t. Finally, observe that for any $u \in t$, the set $\{v \in \mathbb{U} : u * v \in t\}$ is also a plane tree. We denote it by $\theta_u t$ and call it the *(plane) subtree of t stemming from u*. When $u \notin t$, we set $\theta_u t$ to be the empty set by convention (note that this is not a plane tree because it does not contain the empty word $\varnothing$). Subtrees $\theta_u t$ will play the essentially the same role in the remainder of the paper that the subtrees $(T, \varnothing)_{u\downarrow}$ played in Section 1.

We say a plane tree t is *d-ary* if $\mathrm{k}_u(t) \in \{0, d\}$ for all $u \in t$. If t is a d-ary plane tree with m leaves then $|t \setminus \{\varnothing\}| = \sum_{u \in t} \mathrm{k}_u(t) = (|t| - m)d$, so $(d-1)|t| = dm - 1$.

A plane tree t can be naturally viewed as a graph in the following way. Each word $u \in t$ represents a node in the graph, and the edge set consists of the pairs $\{\overleftarrow{u}, u\}$, where $u \in t \setminus \{\varnothing\}$. This clearly yields a connected and acyclic graph, i.e. a tree. By a slight abuse of notation, we identify the plane tree t with its associated graph, which we also denote by t. Note that in the graph-theoretic sense, in a d-ary tree, all non-leaf nodes aside from the root have $d+1$ neighbours (d children and a parent).
Observe that for any $v \in t \setminus \{\varnothing\}$, removing the edge $\{\overleftarrow{v}, v\}$ cuts the tree into two connected components, one of them being $\{w \in t : v \preceq w\}$. Depending on whether or not v is an ancestor of u, we can relate either $(t, u)_{v\downarrow}$ or $(t, u)_{\overleftarrow{v}\downarrow}$ to $\theta_v t$. More precisely, we find that

- if v is not an ancestor of u, then the node set of $(t, u)_{v\downarrow}$ is $\{w \in t : v \preceq w\}$, and so $|(t, u)_{v\downarrow}| = |\theta_v t|$;
- if v is an ancestor of u, then the node set of $(t, u)_{\overleftarrow{v}\downarrow}$ is the complement in t of $\{w \in t : v \preceq w\}$, and so $|(t, u)_{\overleftarrow{v}\downarrow}| = |t| - |\theta_v t|$.

Therefore, for a plane tree t and $u \in t$, the expression (1.1) of the centrality measure becomes

$$\varphi_t(u) = \prod_{v \in t, v \not\preceq u} |\theta_v t| \cdot \prod_{v \in t, \varnothing \prec v \preceq u} (|t| - |\theta_v t|) \tag{2.2}$$

It follows from (2.2) that for $u \neq \varnothing$, $\varphi_t(u)$ and $\varphi_t(\overleftarrow{u})$ are related by the identity

$$\varphi_t(u)|\theta_u t| = (|t| - |\theta_u t|)\varphi_t(\overleftarrow{u}), \tag{2.3}$$

which will be useful below. Repeatedly applying (2.3) yields that for any vertices $u, v \in t$ with $u \preceq v$,

$$\frac{\varphi_t(u)}{\varphi_t(v)} = \prod_{u \prec w \preceq v} \frac{|\theta_w t|}{|t| - |\theta_w t|}. \tag{2.4}$$

In the preceding equation, and below, we omit the constraint "$w \in t$" from the subscript of the product for succinctness, whenever the tree t is clear from context.
Finally, recall the *competitive ratio* of t defined in Section 1.1,

$$\Phi(t) = \frac{\varphi_t(\varnothing)}{\min_{u \in t} \varphi_t(u)}, \tag{2.5}$$

a quantity which naturally arises in our approach to bounding the number of nodes of t that are more central than the root.

2.2. The centrality measure and the competitive ratio

The following lemma gives a necessary condition on the size of subtrees stemming from vertices that are more central than the root.

Lemma 2.2. *For any plane tree t and $u \in t$, if $|\theta_u t|/|t| < (1+\Phi(t))^{-1}$ then $\varphi_t(u) > \varphi_t(\varnothing)$.*

Proof. Note that $\varphi_t(u)|\theta_u t| = (|t| - |\theta_u t|)\varphi_t(\overleftarrow{u})$ by (2.3), so

$$\frac{\varphi_t(\varnothing)}{\varphi_t(u)} \cdot \frac{|t| - |\theta_u t|}{|\theta_u t|} = \frac{\varphi_t(\varnothing)}{\varphi_t(\overleftarrow{u})} \leq \Phi(t)$$

by the definition of Φ. The assumption of the lemma gives $\Phi(t) < (|t| - |\theta_u t|)/|\theta_u t|$, and so $\varphi_t(\varnothing)/\varphi_t(u) < 1$. □

Lemma 2.3. *For a plane tree t, define the sequence of nodes $\varnothing = u_0 \preceq u_1 \preceq \cdots \preceq u_k$ where for each $0 \leq i < k$, u_{i+1} is the unique child of u_i such that $|\theta_{u_{i+1}} t| \geq \frac{1}{2}|t|$ and $k \geq 0$ is the first time such a child does not exist. Then,*

$$\Phi(t) \leq \prod_{i=1}^{k} \frac{1}{1 - |\theta_{u_i} t|/|t|}\,. \tag{2.6}$$

Proof. We first show that u_k minimizes φ_t. Using (2.3), we have that $\varphi_t(\overleftarrow{u})/\varphi_t(u) < 1$ if and only if $|\theta_u t| < |t|/2$, so

$$\varphi_t(\overleftarrow{u}) \geq \varphi_t(u) \quad \iff \quad |\theta_u t| \geq \frac{|t|}{2}\,. \tag{2.7}$$

This relation directly gives us that $\varphi_t(\varnothing) \geq \varphi_t(u_1) \geq \cdots \geq \varphi_t(u_k)$. Next, let w be any child of u_k and v be any descendant of w, and let $u_k \preceq w \preceq \cdots \preceq v$ be the unique path from u_k to v. Then using (2.7) and $|\theta_w t| < |t|/2$, we get $\varphi_t(u_k) < \varphi_t(w) < \cdots < \varphi_t(v)$. Finally, for $0 \leq i < k$, if w is a child of u_i other than u_{i+1}, then $|\theta_w t| < |t|/2$, which implies in the same way as above that any descendant v of w has $\varphi_t(u_i) < \varphi_t(v)$. Combining these results, we readily obtain that u_k minimizes φ_t. Using (2.4), we get

$$\Phi(t) = \frac{\varphi_t(\varnothing)}{\varphi_t(u_k)} = \prod_{i=1}^{k} \frac{|\theta_{u_i} t|}{|t| - |\theta_{u_i} t|}\,.$$

We obtain the desired result after bounding $|\theta_{u_i} t|$ in the numerator above by $|t|$. □

2.3. Preflows and flows on $\mathbb{U}$

We define a *preflow* as a function $p : \mathbb{U} \to [0,1]$ such that $p(\varnothing) = 1$ and $\sum_{i\geq 1} p(u*i) \leq p(u)$ for all $u \in \mathbb{U}$. A *flow* is a preflow for which $\sum_{i\geq 1} p(u*i) = p(u)$ for all $u \in \mathbb{U}$. We further say that a flow (or a preflow) p is *d-ary* when $p(u*j) = 0$ for all $j \geq d+1$ and $u \in \mathbb{U}$. Given a function $f : \mathbb{U} \to [0,\infty)$, for $x > 0$ write

$$E_x(f) := \Big\{ u \in \mathbb{U} : x \prod_{\varnothing \prec v \preceq u} \frac{f(v)}{2} \geq 1 \Big\} \quad \text{and} \quad N_x(f) = |E_x(f)|. \tag{2.8}$$

The next proposition bounds $N_x(\gamma)$ for certain geometrically decaying functions γ; it will later be used to prove upper bounds on the number of nodes which have φ_t-value greater than or equal to the root but which lie in subtrees containing less than ⅓ of all nodes of t, for plane trees $t \subset \mathbb{U}$ and $t \subset \mathbb{U}_d$.

Proposition 2.4. *Let $\alpha \in (1,2]$ and let $\gamma = \gamma_\alpha : \mathbb{U} \to [0,1]$ be inductively defined by $\gamma(\varnothing) = 1$ and $\gamma(u*j) = \alpha^{1-j}\gamma(u)$ for all $u \in \mathbb{U}$ and $j \geq 1$. Then there is a universal constant $c > 0$, that does not depend on α, such that for any $x \geq 1$, we have $N_x(\gamma) \leq \exp(c + c\sqrt{\log_\alpha x})$.*

Proof. Let $u = (u_1,\ldots,u_h) \in \mathbb{U}$ and note that $\gamma(u) = \alpha^{-\sum_{i=1}^h (u_i-1)}$. As the ancestors v of u distinct from $\varnothing$ are the nodes $(u_1,\ldots,u_j)$ with $j \in [h]$, the identity $\sum_{j=1}^h \sum_{i=1}^j (u_i - 1) = \sum_{i=1}^h (h+1-i)(u_i-1)$ gives us that

$$\prod_{\varnothing \prec v \preceq u} \frac{\gamma(v)}{2} = \frac{1}{2^h} \prod_{i=1}^h \alpha^{-(h+1-i)(u_i-1)}. \tag{2.9}$$

Thus, writing $u_i - 1 = j_i$ and taking the logarithm of the product, we obtain that

$$N_x(\gamma) = \left| \left\{ (j_1,\ldots,j_h) \in \mathbb{N}_0^h : h \in \mathbb{N}_0, \log_\alpha(2)h + \sum_{i=1}^h (h+1-i)j_i \leq \log_\alpha x \right\} \right|,$$

where $\mathbb{N}_0 = \{0,1,2,\ldots\}$. Letting $n = \lfloor \log_\alpha x \rfloor$ and reversing the indexing of the j_i, we find the bound

$$N_x(\gamma) \leq \left| \left\{ (j_1,\ldots,j_h) \in \mathbb{N}_0^h : h \leq n, \sum_{k=1}^h k j_k \leq n \right\} \right|.$$

This holds because $\log_\alpha(2) \geq 1$, so we can require that $h \leq n$. Then, observing that for $h \leq n$, the vector $(j_1,\ldots,j_h)$ has the same value of the sum $\sum_{k=1}^h k j_k$ as the vector $(j_1,\ldots,j_h,0,\ldots,0)$ with n coordinates, we can partition the set according to the value of this sum and obtain that

$$N_x(\gamma) \leq \sum_{s=0}^n (n+1) \left| \left\{ (j_1,\ldots,j_n) \in \mathbb{N}_0^n : \sum_{k=1}^n k j_k = s \right\} \right|. \tag{2.10}$$

To conclude the proof, we apply Erdos' non-asymptotic version of the Hardy-Ramanujan formula on the number of partitions of an integer [12], which asserts that for any $s \geq 1$,

$$\left| \left\{ (j_1,\ldots,j_n) \in \mathbb{N}_0^n : \sum_{k=1}^n k j_k = s \right\} \right| \leq \exp\left(\pi\sqrt{2s/3}\right). \tag{2.11}$$

Inserting (2.11) within (2.10) yields that $N_x(\gamma) \leq (n+1)^2 \exp(\pi\sqrt{2n/3})$, which completes the proof. $\square$

2.4. Some distributional definitions and identities

In this section we recall the definitions of Beta and Dirichlet random variables, and some relations between them. We will make frequent use of the standard Gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$. Also, we write $X \sim \mu$ to mean that X has distribution μ, and write $X \preceq_{\mathrm{st}} Y$ if X is stochastically dominated by Y.

For $\alpha, \beta > 0$, say that a random variable B is Beta(α,β)-*distributed* if it has density $x^{\alpha-1}(1-x)^{\beta-1}/B(\alpha,\beta)$ with respect to Lebesgue measure on $[0,1]$; here $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is a normalizing constant. If $B \sim \mathrm{Beta}(\alpha,\beta)$ then

$$\mathbf{E}[B] = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \mathrm{Var}(B) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

If $\beta = (k-1)\alpha$, then this yields that $\mathbf{E}[B^2] = {}^{(\alpha+1)}/_{(k(k\alpha+1))}$, which we will use below. For random variables $B \sim \mathrm{Beta}(\alpha,\beta)$ and $B' \sim \mathrm{Beta}(\alpha',\beta')$, then

$$B' \preceq_{\mathrm{st}} B \text{ if and only if } \alpha \geq \alpha' \text{ and } \beta \leq \beta'\,; \tag{2.12}$$

see e.g. [16], or [1, Theorem 1].

Given $\alpha_1,\ldots,\alpha_k > 0$, a random vector $X = (X_1,\ldots,X_k)$ is Dir$(\alpha_1,\ldots,\alpha_k)$-*distributed* if it has density

$$f(x_1,\ldots,x_k) = \frac{1}{B(\alpha_1,\ldots,\alpha_k)}\prod_{i=1}^k x_i^{\alpha_i-1}$$

with respect to $(k-1)$-dimensional Lebesgue measure on the simplex

$$\Delta_k = \{(x_1,\ldots,x_k) \in [0,1]^k : x_1+\ldots+x_k = 1\}\,;$$

here $B(\alpha_1,\ldots,\alpha_k) = (\prod_{i=1}^k \Gamma(\alpha_i))/\Gamma(\sum_{i=1}^k \alpha_i)$. If $(X_1,\ldots,X_k) \sim \mathrm{Dir}(\alpha_1,\ldots,\alpha_k)$ then it holds that $X_i \sim \mathrm{Beta}(\alpha_i, \sum_{j\neq i}\alpha_j)$ for each $i \in [k]$. We write $\mathrm{Dir}_k(\alpha)$ as shorthand for the symmetric Dirichlet distribution $\mathrm{Dir}(\alpha_1,\ldots,\alpha_k)$ with $\alpha_i = \alpha$ for each $1 \leq i \leq k$. In this case, the density f may be rewritten as

$$f(x_1,\ldots,x_k) = \frac{\Gamma(k\alpha)}{\Gamma(\alpha)^k}\prod_{j=1}^k x_j^{\alpha-1}\,.$$

If $(X_1,\ldots,X_k) \sim \mathrm{Dir}_k(\alpha)$ then $X_i \sim \mathrm{Beta}(\alpha,(k-1)\alpha)$ for $i \in [k]$.

Finally, we say a random variable G is Geometric(p)-distributed if $\mathbb{P}(G = i) = p(1-p)^{i-1}$ for integers $i \geq 1$, and that a random variable U is Uniform$[a,b]$-distributed if its law is the uniform probability distribution on $[a,b]$.

3. Regular trees

This section is devoted to the proof of Theorem 1.2.

3.1. Presentation and asymptotics of the model

We begin by reformulating the d-regular uniform attachment models within the Ulam–Harris formalism as sequences of random plane trees. It will be notationally cleaner to shift the argument by one, so we will hereafter work with $(d+1)$-regular uniform attachment trees for $d \geq 2$.

Fix $d \geq 2$. Let $T_1 = \{\varnothing, 1, \ldots, d, d+1\}$ and $T_{n+1} = T_n \cup \{V_n * 1, \ldots, V_n * d)\}$ for any $n \geq 1$, where, conditionally given $(T_1,\ldots,T_n)$, V_n is a uniformly random leaf of T_n. It is clear that the graph-isomorphism class of T_n has the same law as that of the $\mathrm{UA}_{d+1}(n)$-distributed tree from in the introduction. For the remainder of the section, the notation $\hat{T}_n \sim \mathrm{UA}_{d+1}(n)$ will mean that $\hat{T}_n$ is a random *plane tree* with the same distribution as T_n (and in particular we can write $T_n \sim \mathrm{UA}_{d+1}(n)$).

Note that T_n has exactly $dn+2$ nodes and $(d-1)n+2$ leaves. Moreover, T_n is a subset of $\mathbb{U}_d$ and for any $u \in T_n \setminus \{\varnothing\}$, u has either 0 or d children in T_n. In other words, if $u \in T_n \setminus \{\varnothing\}$ then $\theta_u T_n$ is a d-ary plane tree. Since the size of T_n is deterministic, the formula (2.2) for φ_{T_n} suggests that studying centrality for this model will require a clear understanding of the asymptotic behaviour of the proportion of nodes of T_n lying in the subtree stemming from any fixed node $u \in \mathbb{U}_d$.

We claim that $\lim_{n\to\infty} |\theta_u T_n|/|T_n|$ exists almost surely for any $u \in \mathbb{U}\setminus\{\varnothing\}$. To see this, let $\tau = \inf\{k \in \mathbb{N}_1 : u \in T_k\}$. If $\tau = \infty$ then $|\theta_u T_n| = 0$ for all $n \in \mathbb{N}_1$ and the result is clear. Otherwise, for $n \geq 0$, colour the leaves of $T_{\tau+n}$, either in blue if they are descendants of u, or in red if not. When a leaf V_n is chosen to have d new children, it ceases to be a leaf but its new children take its previous colour. Hence, conditionally given T_τ, the number $M_{\tau+n}$ of leaves of $\theta_u T_{\tau+n}$ evolves as the number of blue balls in a Pólya urn starting with 1 blue ball and $(d-1)\tau+1$ red balls and where at each step, the drawn ball is returned to the urn along with $d-1$ new balls of same colour. Standard results (see e.g. [8, Section 6]) then imply that $\frac{1}{(d-1)n} M_{\tau+n}$ converges almost surely towards a positive random variable. Since $\theta_u T_{\tau+n}$ is a d-ary plane tree we have $(d-1)|\theta_u T_{\tau+n}| = dM_{\tau+n} - 1$, which yields that the following limits exist almost surely for all $u \in \mathbb{U}$ and $v \in \mathbb{U} \setminus \{\varnothing\}$:

$$P_u = \lim_{n\to\infty} \frac{|\theta_u T_n|}{|T_n|} \quad \text{and} \quad D_v = \lim_{n\to\infty} \mathbf{1}_{\{v \in T_n\}} \frac{|\theta_v T_n|}{|\theta_{\overleftarrow{v}} T_n|}. \tag{3.1}$$

Note that D_v is well-defined because if $v \in T_n$ for some $n \geq 1$, then also $\overleftarrow{v} \in T_n$, so P_v and $P_{\overleftarrow{v}}$ are almost surely positive and $D_v \overset{\text{a.s.}}{=} P_v / P_{\overleftarrow{v}}$. This entails the following useful recursive relation

$$\forall u \in \mathbb{U} \setminus \{\varnothing\}, \quad P_u = P_{\overleftarrow{u}} \cdot D_u \quad \text{almost surely.} \tag{3.2}$$

For example, $P_1 = D_1$ because $P_\varnothing = 1$. We also find that $P_{(1,1)} = D_1 D_{(1,1)}$, or that $P_{(2,1,2)} = D_2 D_{(2,1)} D_{(2,1,2)}$. Inductively, we obtain:

$$\text{if} \quad u = (u_1, u_2, \ldots, u_h) \in \mathbb{U}, \quad \text{then} \quad P_u = \prod_{i=1}^{h} D_{(u_1,\ldots,u_i)}. \tag{3.3}$$

We next study the distribution of the family of asymptotic proportions $(P_u)_{u\in\mathbb{U}}$ by describing that of the ratios $(D_u)_{u \in \mathbb{U}\setminus\{\varnothing\}}$. Since T_n is always a subset of $\mathbb{U}_d$, we have $P_u = D_u = 0$ when $u \in \mathbb{U} \setminus \mathbb{U}_d$ so we only need to give the distribution of $(D_u)_{u \in \mathbb{U}_d \setminus \{\varnothing\}}$. Thus, write $\mathbf{D}_\varnothing = (D_1, \ldots, D_d, D_{d+1})$, which has $d+1$ components, and for $u \in \mathbb{U}_d \setminus \{\varnothing\}$, let $\mathbf{D}_u = (D_{u*1}, \ldots, D_{u*d})$, which has d components. Also recall from Section 2.4 that for $\alpha > 0$ and an integer $k \geq 2$, $\mathrm{Dir}_k(\alpha)$ stands for the symmetric Dirichlet distribution of order k with parameter α.

Proposition 3.1. *The random vectors $(\mathbf{D}_u, u \in \mathbb{U}_d)$ are independent. Moreover, $\mathbf{D}_\varnothing \sim \mathrm{Dir}_{d+1}(\frac{1}{d-1})$, and for all $u \in \mathbb{U}_d \setminus \{\varnothing\}$, $\mathbf{D}_u \sim \mathrm{Dir}_d(\frac{1}{d-1})$.*

Proof. We consider a slight variant of the $(d+1)$-regular uniform attachment where, in this new model, the root has the same number of children as all the other non-leaf vertices (and so its degree is smaller by one than those vertices). Namely, let $T'_0 = \{\varnothing\}$, and for $n \geq 0$ let $T'_{n+1} = T'_n \cup \{V'_n * 1, \ldots, V'_n * d\}$, where V'_n is a uniformly random leaf of T'_n conditionally given $(T'_0, \ldots, T'_n)$. Observe that T'_n is a d-ary plane tree. Also note that $T'_1 = \{\varnothing, 1, \ldots, d\}$, and that T'_n has exactly $1 + (d-1)n$ leaves and $1 + dn$ vertices. The arguments used to obtain (3.1) also entail that for all $u \in \mathbb{U}_d \setminus \{\varnothing\}$, the limit

$$D'_u := \lim_{n\to\infty} \mathbf{1}_{\{u \in T'_n\}} \frac{|\theta_u T'_n|}{|\theta_{\overleftarrow{u}} T'_n|} \tag{3.4}$$

exists almost surely. Note that this limit is 0 for any node u which is a descendant of $(d+1)$ since $(d+1) \notin T'_n$ for any $n \geq 0$. We then set $\mathbf{D}'_u = (D'_{u*1}, \dots, D'_{u*d})$ for all $u \in \mathbb{U}_d$. Now, fix $k \in \{d, d+1\}$. We run a Pólya urn starting with balls of colours $1, \dots, k$, one of each colour, where after each draw, we return the drawn ball along with $d-1$ new balls of the colour drawn. For $i \in [k]$, denote by $X_n^{(i)}$ the number of balls of the i'th colour before the n'th draw, so that $X_1^{(i)} = 1$ and $\sum_{i=1}^k X_n^{(i)} = k + (d-1)(n-1)$. Then, it is a standard result of Athreya [2] (see also [8, Section 6] for a modern approach) that

$$\frac{1}{(d-1)n}(X_n^{(1)}, \dots, X_n^{(k)}) \xrightarrow[n\to\infty]{a.s.} (X^{(1)}, \dots, X^{(k)}) \sim \mathrm{Dir}_k\big(\tfrac{1}{d-1}\big). \tag{3.5}$$

Denote by $Y_n^{(i)}$ the number of times a ball of colour i is drawn before the n'th draw; so $X_n^{(i)} = 1 + (d-1)Y_n^{(i)}$ and $Y_1^{(i)} = 0$ for $i \in [k]$, and $\sum_{i=1}^k Y_n^{(i)} = n-1$. Next, let $(T_n^{(1)})_{n\geq 1}, \dots, (T_n^{(k)})_{n\geq 1}$ be k copies of $(T'_n)_{n\geq 1}$. We assume that $(T_n^{(1)})_{n\geq 1}, \dots, (T_n^{(k)})_{n\geq 1}$, and $(X_n^{(1)}, \dots, X_n^{(k)})_{n\geq 1}$ are independent. Then let $\hat{T}_0 = \{\varnothing\}$, and for $n \geq 1$ define

$$\hat{T}_n = \{\varnothing\} \cup \bigcup_{i=1}^k \{i * w \,:\, w \in T^{(i)}_{Y_n^{(i)}}\},$$

so for $n \geq 1$ and $i \in [k]$ we have $\theta_i \hat{T}_n = T^{(i)}_{Y_n^{(i)}}$; it follows that $\theta_i \hat{T}_n$ has $X_n^{(i)}$ leaves. It is then straightforward to check by induction that if $k = d+1$ then $(\hat{T}_n)_{n\geq 1}$ has the same law as $(T_n)_{n\geq 1}$, and that if $k = d$ then $(\hat{T}_n)_{n\geq 1}$ has the same law as $(T'_n)_{n\geq 1}$. Writing $\theta_{i*u}\hat{T}_n = \theta_u \theta_i \hat{T}_n = \theta_u T^{(i)}_{Y_n^{(i)}}$ for any $u \in \mathbb{U}_d \setminus \{\varnothing\}$, then applying (3.5), we deduce that

$$\lim_{n\to\infty} \mathbf{1}_{\{i*u\in\hat{T}_n\}} \frac{|\theta_{i*u}\hat{T}_n|}{|\theta_{i*\overleftarrow{u}}\hat{T}_n|} = \lim_{n\to\infty} \mathbf{1}_{\{u\in T_n^{(i)}\}} \frac{|\theta_u T_n^{(i)}|}{|\theta_{\overleftarrow{u}} T_n^{(i)}|} \quad \text{and} \quad \lim_{n\to\infty} \frac{|\theta_i \hat{T}_n|}{|\hat{T}_n|} = \lim_{n\to\infty} \frac{X_n^{(i)}}{(d-1)n}.$$

By independence of $(X_n^{(1)}, \dots, X_n^{(k)}), (T_n^{(1)}), \dots, (T_n^{(k)})$, it follows from (3.1) and (3.4) that:

- $\mathbf{D}_\varnothing, (\mathbf{D}_{1*u})_{u\in\mathbb{U}_d}, \dots, (\mathbf{D}_{d*u})_{u\in\mathbb{U}_d}, (\mathbf{D}_{(d+1)*u})_{u\in\mathbb{U}_d}$ are independent;
- $\mathbf{D}'_\varnothing, (\mathbf{D}'_{1*u})_{u\in\mathbb{U}_d}, \dots, (\mathbf{D}'_{d*u})_{u\in\mathbb{U}_d}$ are independent.

Since $(T_n^{(i)}) \overset{d}{=} (T'_n)$, it also follows that $(\mathbf{D}_{i*u})_{u\in\mathbb{U}}$ and $(\mathbf{D}'_{i'*u})_{u\in\mathbb{U}_d}$ have the same law as $(\mathbf{D}'_u)_{u\in\mathbb{U}}$ for all $i \in [d+1]$ and $i' \in [d]$. Moreover, (3.5) yields that $\mathbf{D}_\varnothing \sim \mathrm{Dir}_{d+1}(\frac{1}{d-1})$ and that $\mathbf{D}'_\varnothing \sim \mathrm{Dir}_d(\frac{1}{d-1})$. The claimed independence and the fact that $\mathbf{D}_u \sim \mathrm{Dir}_d(\frac{1}{d-1})$ for all $u \neq \varnothing$ then follow by induction. □

For the next results, recall that $\mathrm{ht}(u)$ denotes the height of $u \in \mathbb{U}$.

Corollary 3.2. *For all $v \in \mathbb{U}_d$ with $\mathrm{ht}(v) \geq 2$, $\mathbf{E}[D_v] = 1/d$ and $\mathbf{E}[D_v^2] = 1/(2d-1)$.*

Proof. Recall from Section 2.4 that if $(X_1, \dots, X_k) \sim \mathrm{Dir}_k(\alpha)$, then $X_j \sim \mathrm{Beta}(\alpha, (k-1)\alpha)$ for $j \in [k]$; so $\mathbf{E}X_j = 1/k$ and $\mathbf{E}X_j^2 = (\alpha+1)/(k(k\alpha+1))$. Taking $\alpha = 1/(d-1)$ and $k = d$, the result then follows from Proposition 3.1. □

To conclude this section, we use our description of the asymptotic proportions to show that they uniformly geometrically decay with the height. Combined with Lemma 2.2, this result will imply that, with high probability, the nodes of T_n that are more central than the root must have bounded height; this corresponds to step (I) from the proof of Theorem 1.2 in Section 1.1. The fact that these better candidates belong with high probability to a deterministic finite subset of $\mathbb{U}_d$ will also simplify the asymptotic analysis of the $(d+1)$-regular uniform attachment model, since it allows using only finitely many of the almost sure convergences $|\theta_u T_n|/n \longrightarrow P_u$.

Lemma 3.3. *Fix $d \geq 2$, and for $n \geq 1$ let $T_n \sim \mathrm{UA}_{d+1}(n)$. Then for all integers $m \geq 1$ and for all $\varepsilon \in (0,1)$, it holds that*

$$\limsup_{n\to\infty} \mathbb{P}\big(\exists u \in \mathbb{U}_d \,:\, \mathrm{ht}(u) \geq m, |\theta_u T_n| \geq \varepsilon n\big) \leq \tfrac{3}{2}(d+1)\varepsilon^{-2}\big(\tfrac{2}{3}\big)^m.$$

Proof. Let $L(m,\varepsilon)$ be the lim sup in the statement of the lemma. For $u \in \mathbb{U}_d$, if $\mathrm{ht}(u) \geq m$ then u has a unique ancestor $v \in \mathbb{U}_d$ at height $\mathrm{ht}(v) = m$, and necessarily $|\theta_v T_n| \geq |\theta_u T_n|$. Therefore,

$$L(m,\varepsilon) \leq \limsup_{n\to\infty} \mathbb{P}\left(\exists v \in \mathbb{U}_d \,:\, \mathrm{ht}(v) = m, |\theta_v T_n| \geq \varepsilon n\right) .$$

Since there are only finitely many nodes v with $\mathrm{ht}(v) = m$ in $\mathbb{U}_d$, we obtain that

$$L(m,\varepsilon) \leq \sum_{\substack{v\in\mathbb{U}_d \\ \mathrm{ht}(v)=m}} \limsup_{n\to\infty} \mathbb{P}\left(\frac{1}{n}|\theta_v T_n| \geq \varepsilon\right) .$$

By (3.3), if $v = (v_1,\ldots,v_m) \in \mathbb{U}_d$, then almost surely

$$\frac{|\theta_v T_n|}{n} \longrightarrow P_v = \prod_{i=1}^{m} D_{(v_1,\ldots,v_i)} .$$

By Proposition 3.1, the random variables in the product on the right are independent and all the variables but $D_{(v_1)}$ (which is bounded by 1) are $\mathrm{Beta}(1/(d-1), 1)$-distributed. Since there are exactly $(d+1)d^{m-1}$ nodes $v \in \mathbb{U}_d$ with $\mathrm{ht}(v) = m$, we deduce from the above inequality that

$$L(m,\varepsilon) \leq (d+1)d^{m-1}\mathbb{P}(\tilde{D}_1\tilde{D}_2\cdots\tilde{D}_{m-1} \geq \varepsilon),$$

where the $\tilde{D}_i$ are IID $\mathrm{Beta}(1/(d-1), 1)$-distributed random variables. By applying Markov's inequality to the squares, and then Corollary 3.2, we obtain that

$$L(m,\varepsilon) \leq (d+1)d^{m-1}\varepsilon^{-2}\mathbf{E}[D^2_{(1,1)}]^{m-1} = (d+1)\varepsilon^{-2}\big(\tfrac{d}{2d-1}\big)^{m-1} .$$

The desired inequality follows since $d \geq 2$. □

3.2. Number of competitors in small subtrees

Let t be an arbitrary d-ary plane tree. The goal of this section is to uniformly bound the number of vertices of t that are selected by the algorithm before the actual root, but that belong to $w * \theta_w t$ for some $w \in t$ for which $|\theta_w t|/|t|$ is small. In other words, we want to complete step (II) from the proof of Theorem 1.2 in Section 1.1. Our strategy is to translate the problem into the framework presented in Section 2.3 by encoding all the proportions $|\theta_w t|/|t|$ into a single preflow $p : \mathbb{U} \to [0,1]$. It will then turn out that the number of vertices of $w * \theta_w t$ that are more central than the root in t can be related to the number $N_x(p) = |\{u \in \mathbb{U} \,:\, x\prod_{\varnothing \prec v \preceq u} \frac{p(v)}{2} \geq 1\}|$, defined in (2.8), for some $x > 0$. This motivates the following lemma.

Lemma 3.4. *There is a constant $c_d > 0$ such that for any d-ary preflow p and any $x \geq 1$, it holds that $N_x(p) \leq \exp(c_d + c_d\sqrt{\log x})$.*

Proof. Set $\alpha = d^{1/(d-1)}$, and define $\gamma = \gamma_\alpha : \mathbb{U} \to [0,1]$ by $\gamma(u_1,\ldots,u_h) = \alpha^{-\sum_{i=1}^{h}(u_i-1)}$ as in Proposition 2.4. Without loss of generality, we can assume that $p(u*1) \geq p(u*2) \geq \ldots \geq p(u*d)$ for any $u \in \mathbb{U}$. We claim that $p(u) \leq \gamma(u)$ for any $u \in \mathbb{U}$. To see this, first note that if $k \geq d+1$ then $p(u*k) = 0 \leq \gamma(u*k)$. If $k \in [d]$ then we have

$$p(u*k) \leq \frac{1}{k}\sum_{j=1}^{k} p(u*j) \leq \frac{p(u)}{k} ,$$

since p is a preflow. The choice of α yields that $\frac{1}{k} \le \alpha^{1-k}$ for $k \in [d]$; to see this note that by taking logs it suffices to show that ${}^{\log s}/(s-1)$ is decreasing in s for $s > 1$, which is straightforward. Thus, $p(u * k) \le \alpha^{1-k} p(u)$, and the claim then follows by induction. Since $p \le \gamma$, we deduce that for any d-ary preflow p, we have $N_x(p) \le N_x(\gamma)$, from which we conclude the proof thanks to Proposition 2.4. $\square$

The next proposition is the key bound on the number of nodes that are more central than the root, lying in a given small subtree. Recall from (2.5) that $\Phi(t) = \frac{\varphi_t(\varnothing)}{\min_{u \in t} \varphi_t(u)}$.

Proposition 3.5. *Let $c_d > 0$ be the constant given in Lemma 3.4. Then for any plane tree t and any node $u \in t$, if $\theta_u t$ is d-ary and $|\theta_u t| \le \frac{1}{3}|t|$ then $\big|\{v \in \theta_u t : \varphi_t(\varnothing) \ge \varphi_t(u*v)\}\big| \le \exp(c_d + c_d\sqrt{\log \Phi(t)})$.*

Proof. Define a d-ary preflow by setting $p(v) = |\theta_{u*v} t|/|\theta_u t|$ for all $v \in \mathbb{U}$. For any $v \in \theta_u t$, using the expression (2.4), we can then write

$$\frac{\varphi_t(u)}{\varphi_t(u*v)} = \prod_{\varnothing \prec w \preceq v} \frac{|\theta_{u*w} t|}{|t| - |\theta_{u*w} t|} = \prod_{\varnothing \prec w \preceq v} \frac{p(w)}{\frac{|t|}{|\theta_u t|} - p(w)}.$$

Under the assumption that $|\theta_u t| \le \frac{1}{3}|t|$, this provides the bound

$$\frac{\varphi_t(u)}{\varphi_t(u*v)} \le \prod_{\varnothing \prec w \preceq v} \frac{p(w)}{2},$$

which, in turn, implies that

$$N_x(p) = \left| \left\{ v \in \mathbb{U} : \prod_{\varnothing \prec w \preceq v} \frac{p(w)}{2} \ge \frac{1}{x} \right\} \right| \ge |\{v \in \theta_u t : \varphi_t(u) x \ge \varphi_t(u*v)\}|$$

for all $x > 0$. Since $\Phi(t) \ge \varphi_t(\varnothing)/\varphi_t(u)$, taking $x = \Phi(t)$ in the above inequality, Lemma 3.4 yields that

$$\big|\{v \in \theta_u t : \varphi_t(\varnothing) \ge \varphi_t(u*v)\}\big| \le \big|\{v \in \theta_u t : \varphi_t(u)\Phi(t) \ge \varphi_t(u*v)\}\big| \le \exp(c_d + c_d\sqrt{\log \Phi(t)})$$

as required. $\square$

3.3. Competitive ratio

To derive a good bound of the number of competitors in a $(d+1)$-regular uniform attachment tree by using Proposition 3.5, we first need to control its competitive ratio. Hence, the goal of this section is to show the result below, which corresponds to step (II) in the proof of Theorem 1.2 in Section 1.1.

Proposition 3.6. *There exist constants $c, C > 0$ such that the following holds. Fix $d \ge 2$, and for $n \ge 1$ let $T_n \sim \mathrm{UA}_{d+1}(n)$. Then $\limsup_{n\to\infty} \mathbb{P}(\Phi(T_n) \ge x) \le C x^{-c}$ for all $x \ge 1$.*

The idea of the proof is to combine the deterministic upper bound given by Lemma 2.3 for the competitive ratio with the asymptotic description of the proportions discussed in Section 3.1. It turns out that for the bound we obtain in this way, the upper tail behaviour can be controlled using the tail bound from the following general lemma.

Lemma 3.7. *For all $\beta \in (0,1)$ and $b \ge 1$ there exist $C, c > 0$ such that the following holds. Let $(V_i)_{i\ge1}$ be independent random variables supported by $[0,1]$ such that $\mathbf{E}[(1-V_i)^{-\beta}] \le b$ for all $i \ge 1$. Let $G = \inf\{i \ge 1 : V_1 \cdots V_i \le 1/2\}$ and set*

$$X = \prod_{i=1}^{G-1} \frac{1}{1 - V_1 \cdot \ldots \cdot V_i}. \tag{3.6}$$

Then $\mathbb{P}(X \ge x) \le C x^{-c}$ for all $x \ge 1$.

Proof. First, Markov's inequality yields that for all $i, m \geq 1$,

$$\mathbb{P}\big(V_i \geq 2^{-1/m}\big) = \mathbb{P}\big((1-V_i)^{-\beta} \geq (1-2^{-1/m})^{-\beta}\big) \leq b(1-2^{-1/m})^{\beta}.$$

Thus, there exists an integer $m = m(\beta, b) \geq 1$ such that $\mathbb{P}(V_i \geq 2^{-1/m}) \leq 1/2$ for all $i \geq 1$. Next, for any integer $g \geq 1$ we have

$$\mathbb{P}(X \geq x) \leq \mathbb{P}(G \geq mg+1) + \mathbb{P}\left(\prod_{i=1}^{mg} \frac{1}{1-V_i} \geq x\right) \tag{3.7}$$

because $0 \leq V_j \leq 1$ for any $j \geq 1$. Since $(V_i)_{i\geq 1}$ are independent,

$$\mathbf{E}\left[\left(\prod_{i=1}^{mg} \frac{1}{1-V_i}\right)^{\beta}\right] = \prod_{i=1}^{mg} \mathbf{E}\left[(1-V_i)^{-\beta}\right] \leq b^{mg},$$

so Markov's inequality provides the bound

$$\mathbb{P}\left(\prod_{i=1}^{mg} \frac{1}{1-V_i} \geq x\right) \leq \frac{b^{mg}}{x^{\beta}}. \tag{3.8}$$

By the definition of G, if there exist m distinct indices $i_1 < \ldots < i_m$ such that $V_{i_1}, \ldots, V_{i_m}$ are all less than $2^{-1/m}$, then $G \leq i_m$. Therefore, we can stochastically bound G above by the sum of m independent geometric random variables $Y_1, \ldots, Y_m$ with common parameter $\inf_{i\geq 1} \mathbb{P}(V_i \leq 2^{-1/m})$, which we recall to be at least $1/2$ by our choice of m. It follows that $\mathbb{P}(G \geq mg+1) \leq m\mathbb{P}(Y_1 \geq g+1) \leq m/2^g$ for any $g \geq 0$. Then, we can rewrite (3.7) by using (3.8) and the previous inequality to obtain

$$\mathbb{P}(X \geq x) \leq \frac{m}{2^g} + \frac{b^{mg}}{x^{\beta}}.$$

Finally, choose $c \in (0,1)$ small enough to have that $\beta - cm\log_2 b \geq c$, and let $g = \lfloor c \log_2 x \rfloor$. We then get $\mathbb{P}(X \geq x) \leq (2m+1)x^{-c}$ as desired. □

Since we wish to apply Lemma 3.7 to prove Proposition 3.6, we state and prove a technical estimate which will ensure that the assumptions of Lemma 3.7 are satisfied in the relevant situation. In what follows, the random variables $(D_u)_{u\in\mathbb{U}}$ are as in (3.1).

Proposition 3.8. *Let $V_\varnothing = \max_{i\in[d+1]} D_i$ and $V_v = \max_{i\in[d]} D_{v*i}$ for all $v \in \mathbb{U}_d \setminus \{\varnothing\}$. Then, it holds that $\mathbf{E}[(1-V_u)^{-1/2}] \leq 7\sqrt{2}$ for all $u \in \mathbb{U}_d$.*

Proof. Set $k = d+1$ if $u = \varnothing$ or $k = d$ otherwise, so that V_u is the maximum of the components of a $\mathrm{Dir}_k(\frac{1}{d-1})$-distributed random vector by Proposition 3.1. The components of a symmetric Dirichlet random vector are non-negative and sum to 1, so at most one of them is greater than $1/2$. Since they are identically distributed, we have

$$\begin{aligned}\mathbf{E}\big[(1-V_u)^{-1/2}\big] &\leq \sqrt{2} + k\mathbf{E}\big[\mathbf{1}_{\{D_{u*1}>1/2\}}(1-D_{u*1})^{-1/2}\big] \\ &\leq \sqrt{2} + (d+1)\mathbf{E}\big[\mathbf{1}_{\{D_{u*1}>1/2\}}(1-D_{u*1})^{-1/2}\big],\end{aligned}$$

where we recall from Section 2.4 that D_{u*1} is $\mathrm{Beta}(\frac{1}{d-1}, \frac{k-1}{d-1})$-distributed.
As we have $\frac{k-1}{d-1} \geq 1$, the stochastic domination relation (2.12) implies that $D_{u*1} \preceq_{\mathrm{st}} Y \sim \mathrm{Beta}(\frac{1}{d-1}, 1)$. Since the function $x \in (0,1) \mapsto \mathbf{1}_{\{x>1/2\}}(1-x)^{-1/2}$ is non-decreasing and non-negative, and the normalizing constant in the $\mathrm{Beta}(\frac{1}{d-1}, 1)$ distribution's density is $\Gamma(\frac{1}{d-1})\Gamma(1)/\Gamma(1+\frac{1}{d-1}) = d-1$, it follows that

$$\begin{aligned}\mathbf{E}\big[(1-V_u)^{-1/2}\big] &\leq \sqrt{2} + (d+1)\mathbf{E}\big[\mathbf{1}_{\{Y>1/2\}}(1-Y)^{-1/2}\big] \\ &= \sqrt{2} + \frac{d+1}{d-1}\int_{1/2}^{1} (1-x)^{-1/2} x^{1/(d-1)-1}\,\mathrm{d}x.\end{aligned}$$

Next, we use that $0 \geq 1/(d-1) - 1 \geq -1$ to write

$$\mathbf{E}\big[(1-V_u)^{-1/2}\big] \leq \sqrt{2} + \frac{2(d+1)}{d-1}\int_{1/2}^{1}(1-x)^{-1/2}\,\mathrm{d}x = \sqrt{2}\cdot\frac{3d+1}{d-1}.$$

The desired result follows since $d \geq 2$. □

Proof of Proposition 3.6. Using the random variables $(P_u)_{u\in\mathbb{U}}$ introduced in Section 3.1, define a random path $(u_i)_{i\geq 0}$ in $\mathbb{U}_d$ as follows. For $u \in \mathbb{U}_d$, let $j(u) = \arg\max(D_{u*i}, i \geq 1)$, so that $P_{u*j(u)} = \max_{i\geq 1} P_{u*i} = P_u D_{u*j(u)}$. By Proposition 3.1, the random variables $(D_{u*i})_{i\geq 1}$ almost surely have a unique maximum so $j(u)$ is well-defined almost surely, and almost surely $j(\varnothing) \in [d+1]$ and $j(u) \in [d]$ for $u \neq \varnothing$. Observe that, in the notation of Proposition 3.8, we have $V_u = D_{u*j(u)}$ for all $u \in \mathbb{U}_d$.

Let $u_0 = \varnothing$ and, inductively, for $i \geq 1$, given u_i set $u_{i+1} = u_i * j_{u_i}$. We then have $P_{u_i} = P_{u_{i-1}} D_{u_i} = \prod_{j=1}^{i} D_{u_j}$ for $i \geq 1$. By Proposition 3.1, the random pairs $(D_{u*j(u)}; j(u))_{u\in\mathbb{U}_d}$ are independent and have the same distribution except for $(D_{j_\varnothing}; j(\varnothing))$, which implies that the random variables $(D_{u_i})_{i\geq 1}$ are independent and the random variables $(D_{u_i})_{i\geq 2}$ are identically distributed. Moreover, each D_{u_i} has a continuous distribution and support $[0,1]$, so writing $G = \inf\{k \geq 1 : P_{u_k} \leq 1/2\}$, it follows that G is almost surely finite. It also follows that $\mathbb{P}(\exists k \in \mathbb{N}_1 : \prod_{i\in[k]} D_{u_i} = 1/2) = 0$, so almost surely $P_{u_G} < 1/2$. Since $|\theta_{u_i} T_n|/|T_n| \to P_{u_i}$ almost surely for all $i \geq 1$, it follows that there exists a random variable $n_0 \in \mathbb{N}_1$ such that almost surely, for all $n \geq n_0$, $|\theta_{u_i} T_n| > \frac{1}{2}|T_n|$ for all $i \in [G-1]$ and $|\theta_{u_{G-1}*j} T_n| < \frac{1}{2}|T_n|$ for all $j \in [d]$. It follows that for $n \geq n_0$, $\varnothing = u_0 \preceq \cdots \preceq u_{G-1}$ is the sequence of nodes described in Lemma 2.3. The conclusion of that lemma is that almost surely, for $n \geq n_0$,

$$\Phi(T_n) \leq \prod_{i=1}^{G-1} \frac{1}{1 - |\theta_{u_i} T_n|/|T_n|}.$$

Since $|\theta_{u_i} T_n|/|T_n| \xrightarrow{\text{a.s.}} P_{u_i}$, it follows that $\limsup_{n\to\infty} \Phi(T_n) \leq \prod_{i=1}^{G-1}(1-P_{u_i})^{-1}$ almost surely, so

$$\limsup_{n\to\infty} \mathbb{P}(\Phi(T_n) \geq x) \leq \mathbb{P}\left(\prod_{i=1}^{G-1} \frac{1}{1-P_{u_i}} \geq x\right) = \mathbb{P}\left(\prod_{i=1}^{G-1} \frac{1}{1-D_{u_1}\cdot\ldots\cdot D_{u_i}} \geq x\right).$$

The random variables $(D_{u_i})_{i\geq 1} = (V_{u_{i-1}})_{i\geq 1}$ are independent as observed above, and by Proposition 3.8 we have $\mathbf{E}[(1-D_{u_i})^{-1/2}] \leq 7\sqrt{2}$ for all $i \geq 1$. The result then follows from the above displayed equation and the bound of Lemma 3.7. □

3.4. The dependence on d in Theorem 1.2

By examining the proof given in Section 1.1, we see that the constant c_d^* appearing inside the exponential in Theorem 1.2 depends on d exclusively via the constant c_d from Lemma 3.4 and Proposition 3.5. However, one cannot remove the dependence in d in Lemma 3.4; it is straightforward to show that if $p(\varnothing) = 1$ and $p(u*i) = \mathbf{1}_{\{i\leq d\}}\frac{p(u)}{d}$ for all $u \in \mathbb{U}$ and $i \geq 1$, then p is a d-ary flow satisfying $\log N_x(p) \sim \sqrt{2(\log d)\log x}$ as $x \to \infty$. Thus, deterministic arguments are not enough to uniformly control the optimal constants c_d^*, or to bound from below the minimum size ensuring a given precision of the root-finding algorithm for random trees with unbounded degrees. Therefore, in Section 4, when proving Theorem 1.1, we are forced to rely more heavily on the distributional properties of the uniform attachment model.

It turns out that similar arguments to those of Section 4 can also be used to show that the d-dependence of the constants c_d^* from the exponential term in the bound of Theorem 1.2 can be removed. While we will not write a complete proof of this fact in the paper in order to avoid too much repetition, we will explain the arguments required for this adaptation later, in Section 4.5.

4. Uniform attachment trees

This section is devoted to the proof of Theorem 1.1.

4.1. Presentation and asymptotics of the model

We start by reformulating the uniform attachment model for plane trees using the Ulam–Harris formalism. Recall from Definition 2.1 that if t is a plane tree and $u \in t$, then $\mathrm{k}_u(t)$ stands for the number of children of u in t. We let $T_1 = \{\varnothing\}$ and for any $n \geq 1$, $T_{n+1} = T_n \cup \{V_n * (\mathrm{k}_{V_n}(T_n) + 1)\}$ where V_n is uniformly random on T_n conditionally given $(T_1, \ldots, T_n)$. By construction, the graph-isomorphism class of T_n has the same law as that of $\mathrm{UA}(n)$-distributed tree presented in the introduction. Throughout this section, we slightly abuse notation by writing $\hat{T}_n \sim \mathrm{UA}(n)$ to mean that $\hat{T}_n$ is a random *plane tree* with the same law as that of the plane tree T_n constructed in this paragraph. This in particular allows us to write $T_n \sim \mathrm{UA}(n)$.

It is well-known, and straightforwardly proved by first and second moment computations, that in a uniform attachment tree the degree of any fixed node tends to ∞ as $n \to \infty$. For $u \in \mathbb{U}$ writing $\tau = \tau_u = \inf\{k \geq 1 : u \in T_k\}$, so that $T_\tau = T_{\tau-1} \cup \{u\}$, it follows that τ_u is almost surely finite for all $u \in \mathbb{U}$; we will use this below.

Note that $|T_n| = n$ for any $n \geq 1$. As we did in Section 3.1 for the d-regular uniform attachment model, we describe the asymptotic behaviour of the proportions of the subtrees of T_n. First, we show that for any $u \in \mathbb{U}$ and any integer $i \geq 1$, the two limits

$$P_u = \lim_{n\to\infty} \frac{|\theta_u T_n|}{n} \quad \text{ and } \quad U_{u*i} = \lim_{n\to\infty} \mathbf{1}_{\{u*i \in T_n\}} \frac{|\theta_{u*i} T_n|}{\sum_{j\geq i} |\theta_{u*j} T_n|} \tag{4.1}$$

exist almost surely. To do this, fix $u \in \mathbb{U} \setminus \{\varnothing\}$ and write $\tau = \tau_u$. Colour u in blue and all the vertices of $T_{\tau-1}$ in red, then colour each node added after time τ in the same colour as its parent. Hence, conditionally given T_τ, $(|\theta_u T_{\tau+n}|)_{n\geq 0}$ evolves as the number of blue balls in a standard Pólya urn that initially contains 1 blue ball and $\tau - 1$ red balls, which classically implies that $\frac{1}{\tau+n}|\theta_u T_{\tau+n}|$ converges towards a nonzero limit almost surely. This shows that P_u is well-defined and that a.s. $P_u > 0$. It follows that the limit U_{u*i} also exists almost surely, and that $U_{u*i} \overset{\text{a.s.}}{=} P_{u*i}(P_u - \sum_{j=1}^{i-1} P_{u*j})^{-1}$. From there, an induction on $i \geq 1$ yields the following recursive relation:

$$\forall u \in \mathbb{U}, \forall i \geq 1, \quad P_{u*i} = P_u \cdot U_{u*i} \prod_{j=1}^{i-1} (1 - U_{u*j})\,. \tag{4.2}$$

For example, we get $P_1 = U_1$, $P_2 = (1-U_1)U_2$, $P_3 = (1-U_1)(1-U_2)U_3$, and $P_{(1,1,2)} = U_1 U_{(1,1)}(1-U_{(1,1,1)})U_{(1,1,2)}$. By induction on u, (4.2) allows us to express all the asymptotic proportions $(P_u)_{u\in\mathbb{U}}$ in terms of the uniforms $(U_v)_{v\in\mathbb{U}\setminus\{\varnothing\}}$ as follows:

$$\text{if} \quad u = (u_1, u_2, \ldots, u_k), \quad \text{ then } \quad P_u = \prod_{i=1}^{k} \left(U_{(u_1,\ldots,u_i)} \prod_{j=1}^{u_i - 1} (1 - U_{(u_1,\ldots,u_{i-1},j)}) \right). \tag{4.3}$$

Similarly as in Section 3.1, we can describe the joint law of the factors $(U_u)_{u\in\mathbb{U}\setminus\{\varnothing\}}$.

Proposition 4.1. *The random variables $(U_u)_{u\in\mathbb{U}\setminus\{\varnothing\}}$ are independent and Uniform$[0,1]$-distributed.*

Proof. Let $(X_n)_{n\geq 2}$ denote the number of blue balls at time n in a Pólya urn that contains 1 blue ball and 1 red ball at time 2, so that $X_2 = 1$. Let also $(T'_n)_{n\geq 1}$ and $(T''_n)_{n\geq 1}$ be two sequences of plane trees that both follow the uniform attachment model, i.e., they have the same law as $(T_n)_{n\geq 1}$. We assume that $(X_n)_{n\geq 2}$, $(T'_n)_{n\geq 1}$, and $(T''_n)_{n\geq 1}$ are independent. Then, for all $u\in\mathbb{U}$ and $i\geq 1$, let

$$U_{u*i}=\lim_{n\to\infty}\mathbf{1}_{\{u*i\in T'_n\}}\frac{|\theta_{u*i}T'_n|}{\sum_{j\geq i}|\theta_{u*j}T'_n|}\quad\text{and}\quad U''_{u*i}=\lim_{n\to\infty}\mathbf{1}_{\{u*i\in T''_n\}}\frac{|\theta_{u*i}T''_n|}{\sum_{j\geq i}|\theta_{u*j}T''_n|}$$

be defined from (T'_n) and (T''_n) as U_u is defined from (T_n) in (4.1).

We now inductively construct another growing sequence $(\hat{T}_n)_{n\geq 1}$ of plane trees. Set $\hat{T}_1=\{\varnothing\}$ and $\hat{T}_2=\{\varnothing,1\}$. Then for any $n\geq 2$, define $\hat{T}_{n+1}$ as follows:

- if $X_{n+1}=X_n+1$ then $\hat{T}_{n+1}=\hat{T}_n\cup\{1*w'\}$, where $w'\in\mathbb{U}$ is the only node of T'_{X_n+1} that is not in T'_{X_n};
- if $X_{n+1}=X_n$ then $\hat{T}_{n+1}=\hat{T}_n\cup\{(j+1)*w''\}$, where $j\geq 1$ and $w''\in\mathbb{U}$ are such that $j*w''$ is the only node of T''_{n-X_n+1} that is not in T''_{n-X_n}.

For any $n\geq 2$, observe that by construction $|\theta_1\hat{T}_n|=X_n$ and $\theta_1\hat{T}_n=T'_{X_n}$, and that $\theta_{1+j}\hat{T}_n=\theta_jT''_{n-X_n}$ for all $j\geq 1$. This allows us to check by induction that $(\hat{T}_n)_{n\geq 1}$ follows the uniform attachment model, and thus we can assume without loss of generality that $\hat{T}_n=T_n$ for all $n\geq 1$. Then, for any $v\in\mathbb{U}$, we can write $\theta_{1*v}T_n=\theta_v\theta_1T_n=\theta_vT'_{X_n}$ and similarly $\theta_{(1+j)*v}T_n=\theta_{j*v}T''_{n-X_n}$ for all $j\geq 1$. Since we know from standard properties of Pólya urns that $X_n\to\infty$ and $n-X_n\to\infty$ almost surely, these identities and the expressions (4.1) entail that for all $u\in\mathbb{U}\setminus\{\varnothing\}$, $v\in\mathbb{U}$, and $j\geq 1$, we have

$$U_{1*u}=U'_u,\quad U_{(1+j)*v}=U''_{j*v},\quad U_1=\lim_{n\to\infty}\tfrac{1}{n}X_n\,.$$

By independence of (X_n), (T'_n), and (T''_n), it follows that the families U_1, $(U_{1*u})_{u\in\mathbb{U}\setminus\{\varnothing\}}$, and $(U_{(1+j)*v})_{j\geq 1,v\in\mathbb{U}}$ are independent. Since $(T_n)\overset{d}{=}(T'_n)\overset{d}{=}(T''_n)$, it also follows that $(U_{1*u})_{u\in\mathbb{U}\setminus\{\varnothing\}}$ has the same law as $(U_u)_{u\in\mathbb{U}\setminus\{\varnothing\}}$, and $(U_{(1+j)*v})_{j\geq 1,v\in\mathbb{U}}$ has the same law as $(U_{j*v})_{j\geq 1,v\in\mathbb{U}}$. Moreover, it is a well-known fact about Pólya urns that $U_1=\lim_{n\to\infty}X_n/n$ is Uniform$[0,1]$-distributed. An inductive argument concludes the proof. □

Recall from (2.1) that for $u=(u_1,\ldots,u_i)\in\mathbb{U}$, the weight of u is $\mathrm{w}(u)=\sum_{j=1}^i u_j$ which we will consider instead of the height $\mathrm{ht}(u)$. Building on Lemma 2.2, we next show that with high probability, there are only finitely many nodes that are more central than the root in a infinite plane tree generated by the uniform attachment model. This is because the weights $\mathrm{w}(u)$ of such nodes must stay bounded. Moreover, thanks to this fact, we will avoid some technical issues by considering only finitely many convergences at a time.

Lemma 4.2. *For any $n\geq 1$, let $T_n\sim\mathrm{UA}(n)$. Then for all integers $m\geq 1$ and for all $\varepsilon\in(0,1)$, it holds that*

$$\limsup_{n\to\infty}\mathbb{P}\big(\exists u\in\mathbb{U}\,:\,\mathrm{w}(u)\geq m,|\theta_uT_n|\geq\varepsilon n\big)\leq\varepsilon^{-2}(2/3)^{m-1}.$$

Proof. We define $L(m,\varepsilon)$ to be the lim sup in the statement of the lemma. Notice that for $u\in\mathbb{U}$, if $\mathrm{w}(u)\geq m$, then there exists a node v and an integer $k\geq 1$ such that $\mathrm{w}(v*k)=m$, and that $v*k$ is an older sibling of u or of some ancestor of u. This gives us that $\sum_{i=k}^\infty|\theta_{v*i}T_n|\geq|\theta_uT_n|$, hence

$$L(m,\varepsilon)\leq\limsup_{n\to\infty}\mathbb{P}\left(\exists v\in\mathbb{U},k\geq 1\,:\,\mathrm{w}(v*k)=m,\sum_{i=k}^\infty|\theta_{v*i}T_n|\geq\varepsilon n\right).$$

Since there are only finitely many nodes u with $\mathrm{w}(u)=m$, we obtain that

$$L(m,\varepsilon)\leq\sum_{\substack{v\in\mathbb{U},k\geq 1\\ \mathrm{w}(v*k)=m}}\limsup_{n\to\infty}\mathbb{P}\left(\sum_{i=k}^\infty\frac{|\theta_{v*i}T_n|}{n}\geq\varepsilon\right).$$

We can rewrite the sum inside the probability as $\frac{1}{n}(|\theta_v T_n| - 1 - \sum_{i=1}^{k-1} |\theta_{v*i} T_n|)$. Letting $n \to \infty$, this random variable converges almost surely to $P_v - \sum_{i=1}^{k-1} P_{v*i}$, which is equal to $P_v \prod_{i=1}^{k-1}(1 - U_{v*i})$ using (4.2) and a telescoping argument. Moreover, by (4.3), note that $P_v \prod_{i=1}^{k-1}(1 - U_{v*i})$ is a product of $\mathrm{w}(v) + k - 1 = \mathrm{w}(v * k) - 1 = m - 1$ independent Uniform$[0, 1]$-distributed random variables. By the Portmanteau theorem we can thus rewrite the above inequality as

$$L(m, \varepsilon) \le \big|\{u \in \mathbb{U} : \mathrm{w}(u) = m\}\big| \cdot \mathbb{P}(U_1 U_2 \cdots U_{m-1} \ge \varepsilon),$$

where the U_i are independent Uniform$[0, 1]$-distributed random variables. It is straightforward to verify by induction that there are exactly 2^{m-1} nodes u with $\mathrm{w}(u) = m$, so we can further rewrite the inequality as

$$L(m, \varepsilon) \le 2^{m-1}\mathbb{P}(U_1 U_2 \cdots U_{m-1} \ge \varepsilon) = 2^{m-1}\mathbb{P}((U_1 U_2 \cdots U_{m-1})^2 \ge \varepsilon^2).$$

Using Markov's inequality and independence, we obtain that

$$L(m, \varepsilon) \le \frac{2^{m-1}}{\varepsilon^2}\mathbf{E}[U_1^2]^{m-1} = \frac{2^{m-1}}{\varepsilon^2 3^{m-1}} . \qquad \square$$

4.2. Number of competitors in small subtrees

The random variables $(P_u)_{u\in\mathbb{U}}$ define a (random) flow on $\mathbb{U}$; however, it is somewhat complex to work with, so we bound it from above by another, better-behaved family of random variables $(Q_u)_{u\in\mathbb{U}}$. The upper bound will be useful as will enforce an exponential decay in the values of "younger siblings", which will be simpler than but similar to the expected behaviour of $(P_u)_{u\in\mathbb{U}}$. We then use the bound to prove the main result of the section, a probabilistic analogue of Lemma 3.4 which establishes an exponential upper tail bound for the random variable $N_x(P) = |E_x(P)| = |\{u \in \mathbb{U} : x \prod_{\varnothing \prec v \preceq u} \frac{P_v}{2} \ge 1\}$.

We begin with a technical lemma; we will then immediately use the lemma to construct the family of random variables $(Q_u)_{u\in\mathbb{U}}$.

Lemma 4.3. *Let $(U_i)_{i\ge1}$ be independent Uniform$[0, 1]$-distributed random variables. Then there exists a sequence $(V_i)_{i\ge1}$ of independent Uniform$[1/2, 1]$-distributed random variables and a random bijection $\sigma : \mathbb{N}_1 \to \mathbb{N}_1$ such that with $P_i = U_i \cdot \prod_{j=1}^{i-1}(1 - U_j)$, then almost surely $P_{\sigma(i)} \le \frac{1}{2}\prod_{j=1}^{i-2} V_j$ for all $i \ge 2$.*

Proof. First, let $K = \inf\left\{i \ge 1 : U_i > \frac{1}{2}\right\}$ which is almost surely finite. Then for $i \ge 1$ set

$$V_i = \begin{cases} 1 - U_i & \text{if } i \le K - 1, \\ 1 - \frac{U_{i+1}}{2} & \text{if } i \ge K, \end{cases}$$

and define $\sigma : \mathbb{N}_1 \to \mathbb{N}_1$ by

$$\sigma(i) = \begin{cases} K & \text{if } i = 1, \\ i - 1 & \text{if } 2 \le i \le K, \\ i & \text{if } i \ge K + 1. \end{cases}$$

For $2 \le i \le K$, it holds that $U_{i-1} \le \frac{1}{2}$ by construction, so

$$P_{\sigma(i)} = P_{i-1} = U_{i-1}\prod_{j=1}^{i-2}(1 - U_j) \le \frac{1}{2}\prod_{j=1}^{i-2} V_j .$$

Also, for $i \ge K + 1$, since $U_K > 1/2$, we have that

$$P_{\sigma(i)} = P_i = U_i\left(\prod_{j=1}^{K-1}(1 - U_j)\right)(1 - U_K)\left(\prod_{j=K+1}^{i-1}(1 - U_j)\right)$$

$$\leq \left(\prod_{j=1}^{K-1} V_j\right) \cdot \frac{1}{2} \cdot \left(\prod_{j=K+1}^{i-1} V_{j-1}\right).$$

Hence, for all $i \geq 2$, we indeed have that $P_{\sigma(i)} \leq \frac{1}{2}\prod_{j=1}^{i-2} V_j$ almost surely.

We now show that the V_i's are independent and Uniform$[1/2,1]$-distributed. Let $i \geq 1$, let $f_1,\dots,f_i : \mathbb{R} \to \mathbb{R}$ be bounded measurable functions, and let $W \sim$ Uniform$[1/2,1]$. It suffices to show that $\mathbf{E}\left[\prod_{j=1}^{i} f_j(V_j)\right] = \prod_{j=1}^{i} \mathbf{E}[f_j(W)]$. We write the left-hand side as a sum over every possible value of K using indicators:

$$\mathbf{E}\left[\prod_{j=1}^{i} f_j(V_j)\right] = \sum_{k\geq 1} \mathbf{E}\left[\left(\prod_{j=1}^{k-1} \mathbf{1}_{\{U_j \leq \frac{1}{2}\}} f_j(1-U_j)\right) \cdot \mathbf{1}_{\{U_k > \frac{1}{2}\}} \cdot \left(\prod_{j=k}^{i} f_j\left(1 - \frac{U_{j+1}}{2}\right)\right)\right].$$

Notice that each factor in the product on the right is a function of a different random variable, so since the $(U_i)_{i\geq 1}$ are IID, we obtain

$$\mathbf{E}\left[\prod_{j=1}^{i} f_j(V_j)\right] = \sum_{k\geq 1} \left(\prod_{j=1}^{k-1} \mathbf{E}\left[\mathbf{1}_{\{U_1 \leq \frac{1}{2}\}} f_j(1-U_1)\right]\right) \cdot \mathbb{P}\big(U_1 > \tfrac{1}{2}\big) \cdot \left(\prod_{j=k}^{i} \mathbf{E}\left[f_j(1-U_1/2)\right]\right).$$

Finally, we observe that W has the same law as $1 - U_1/2$, and that the conditional law of $1 - U_1$ given that $U_1 \leq 1/2$ is also Uniform$[1/2,1]$-distributed. This entails that $\mathbf{E}\left[\prod_{j=1}^{i} f_j(V_j)\right] = \sum_{k\geq 1} 2^{-k} \prod_{j=1}^{i} \mathbf{E}[f_j(W)]$ which implies the desired identity. ☐

Corollary 4.4. *There exists a family $(V_u)_{u\in\mathbb{U}\setminus\{\varnothing\}}$ of* IID *Uniform$[1/2,1]$-distributed random variables and a random bijection $\sigma : \mathbb{U} \to \mathbb{U}$ such that $\sigma(\varnothing) = \varnothing$ and $\sigma(\overleftarrow{u}) = \overleftarrow{\sigma(u)}$ for all $u \in \mathbb{U} \setminus \{\varnothing\}$, and $P_{\sigma(u)} \leq Q_u$ for all $u \in \mathbb{U}$ almost surely, where $(Q_u)_{u\in\mathbb{U}}$ is inductively defined by*

- $Q_\varnothing = 1$;
- $Q_{u*1} = Q_u$ *for all* $u \in \mathbb{U}$;
- $Q_{u*i} = Q_u \cdot \frac{1}{2}\prod_{j=1}^{i-2} V_{u*j}$ *for all* $i \geq 2$ *and all* $u \in \mathbb{U}$.

Proof. For $u \in \mathbb{U}$ write $\mathbf{U}_u = (U_{u*i})_{i\geq 1}$. From Proposition 4.1 we have that the random vectors $(\mathbf{U}_u)_{u\in\mathbb{U}}$ are independent, and within each vector the entries $(U_{u*i})_{i\geq 1}$ are IID and Uniform$[0,1]$-distributed. For each $u \in \mathbb{U}$, Lemma 4.3 thus gives us a sequence $(V'_{u*i})_{i\geq 1}$ of independent Uniform$[1/2,1]$-distributed random variables and a bijection $\sigma_u : \mathbb{N}_1 \to \mathbb{N}_1$ such that $U_{u*\sigma_u(i)} \prod_{j=1}^{\sigma_u(i)-1}(1-U_{u*j}) \leq \frac{1}{2}\cdot\prod_{j=1}^{i-2} V'_{u*j}$ for all $i \geq 2$ almost surely. Moreover, the independence of the vectors $(\mathbf{U}_u)_{u\in\mathbb{U}}$ implies that the sequences $(\sigma_u(i), V'_{u*i})_{i\geq 1}$ for $u \in \mathbb{U}$ can be chosen to be independent.

Now, let us define $\sigma : \mathbb{U} \to \mathbb{U}$ inductively by setting $\sigma(\varnothing) = \varnothing$ and $\sigma(u*i) = \sigma(u) * \sigma_{\sigma(u)}(i)$ for all $u \in \mathbb{U}$ and $i \geq 1$, so that $\overleftarrow{\sigma(u)} = \sigma(\overleftarrow{u})$ for all $u \in \mathbb{U} \setminus \{\varnothing\}$. It is then easy to check that σ is a bijection on $\mathbb{U}$ which preserves the height, i.e. $\mathrm{ht}(\sigma(u)) = \mathrm{ht}(u)$ for all $u \in \mathbb{U}$. Furthermore, for any $h \geq 0$, the restriction of σ to $\{v \in \mathbb{U} : \mathrm{ht}(v) \leq h\}$ is measurable with respect to the family $(\sigma_v(i), V'_{v*i}\,;\, i \geq 1, v \in \mathbb{U}, \mathrm{ht}(v) \leq h-1)$. Setting $V_{u*i} = V'_{\sigma(u)*i}$ for all $u \in \mathbb{U}$ and $i \geq 1$, it follows via induction on the height that the random variables $(V_u)_{u\in\mathbb{U}\setminus\{\varnothing\}}$ are independent Uniform$[1/2,1]$-distributed random variables as desired.

From (4.2), for all $u \in \mathbb{U}$ and $i \geq 2$, we see that $P_{\sigma(u*1)} = P_{\sigma(u)*\sigma_{\sigma(u)}(1)} \leq P_{\sigma(u)}$ and that

$$P_{\sigma(u*i)} = P_{\sigma(u)*\sigma_{\sigma(u)}(i)} \leq P_{\sigma(u)} \cdot \frac{1}{2} \cdot \prod_{j=1}^{i-2} V'_{\sigma(u)*j} = P_{\sigma(u)} \cdot \frac{1}{2} \cdot \prod_{j=1}^{i-2} V_{u*j} = P_{\sigma(u)} \frac{Q_{u*i}}{Q_u}.$$

A straightforward induction then yields that $P_{\sigma(u)} \leq Q_u$ for all $u \in \mathbb{U}$ almost surely. ☐

By contrast with regular trees, in the uniform attachment case, we cannot invoke a deterministic bound such as Proposition 3.5. Nonetheless, we can provide a probabilistic replacement of Proposition 3.5 that relies on the distribution of the family $P = (P_u)_{u\in\mathbb{U}}$ by using Corollary 4.4. This result represents the main goal of the present section and will be crucial to prove Theorem 1.1 later. Recall from (2.8) that for $f : \mathbb{U} \to [0, \infty)$ and $x > 0$, we write $E_x(f) = \big\{u \in \mathbb{U} : x \prod_{\varnothing\prec v\preceq u} \frac{f(v)}{2} \geq 1\big\}$ and $N_x(f) = |E_x(f)|$.

Proposition 4.5. *There exists a constant $c > 0$ such that for any $x \geq 1$ and $y \geq 1$,*

$$\mathbb{P}\big(N_x(P) \geq y \exp(c + c\sqrt{\log x})\big) \leq 2e^{-y}.$$

We now introduce some notation and prove a technical lemma that will be used to show Proposition 4.5. In what follows, we use the function $\gamma = \gamma_{4/3}$ defined in Proposition 2.4; recall this function is given by $\gamma(u) = (4/3)^{-\sum_{i=1}^{h}(u_i-1)}$ for $u = (u_1, \dots, u_h) \in \mathbb{U}$. Also, for $u = (u_1, \dots, u_h) \in \mathbb{U}$, let $\mathrm{r}(u) = \sum_{i=1}^{l} \mathbf{1}_{\{u_i\geq 2\}}$ be the number of nodes on the ancestral path of u which are not oldest children.

Lemma 4.6. *There is a constant $c > 0$ such that $\sup_{u\in E_x(\gamma)} \mathrm{r}(u) \leq c\sqrt{\log x}$ for all $x \geq 1$.*

Proof. Fix $u = (u_1, \dots, u_h) \in \mathbb{U}$. Note that if i is the k'th index for which $u_i \geq 2$, then we have $h+1-i \geq \mathrm{r}(u)+1-k$. This entails that $\sum_{i=1}^{h}(h+1-i)(u_i-1) \geq \sum_{k=1}^{\mathrm{r}(u)}(\mathrm{r}(u)+1-k) = \frac{1}{2}\mathrm{r}(u)(\mathrm{r}(u) + 1)$. Therefore, it follows from the identity (2.9) with $\alpha = 4/3$ that

$$\prod_{\varnothing\prec v\preceq u} \frac{\gamma(v)}{2} = \frac{1}{2^h}\prod_{i=1}^{h}\left(\frac{3}{4}\right)^{(h+1-i)(u_i-1)} \leq \left(\frac{3}{4}\right)^{\mathrm{r}(u)(\mathrm{r}(u)+1)/2},$$

so if $u \in E_x(\gamma)$ then $(4/3)^{\mathrm{r}(u)^2/2} \leq x$, and thus $\mathrm{r}(u) \leq \sqrt{2\log_{4/3} x}$. The result follows. □

Proof of Proposition 4.5. Our strategy for bounding $N_x(P)$ is to compare it with $N_x(\gamma)$, where $\gamma = \gamma_{4/3}$ is as above and in Proposition 2.4. Let $Q = (Q_u)_{u\in\mathbb{U}}$ and $\sigma : \mathbb{U} \to \mathbb{U}$ be as in Corollary 4.4. From the identity $\sigma(\overleftarrow{u}) = \overleftarrow{\sigma(u)}$, we observe that the ancestors of $\sigma(u)$ are the nodes $\sigma(v)$ with $v \preceq u$. Since $P_{\sigma(u)} \leq Q_u$, it follows that if $\sigma(u) \in E_x(P)$ then $u \in E_x(Q)$, so $N_x(P) \leq N_x(Q)$ almost surely.

For $v \in \mathbb{U}$, let $\tau_{v,1} = 0$ and, for all $i \geq 2$, let $\tau_{v,i} = \inf\{j > \tau_{v,i-1} : V_{v*j} \leq \frac{3}{4}\}$, where $(V_u)_{u\in\mathbb{U}\setminus\{\varnothing\}}$ are given by Corollary 4.4. Note that the increments $\tau_{v,i} - \tau_{v,i-1}$ are Geometric(1/2)-distributed. Moreover, it follows from the independence of the random variables $(V_u)_{u\in\mathbb{U}\setminus\{\varnothing\}}$ that the increments $(\tau_{v,i} - \tau_{v,i-1}\,;\, v \in \mathbb{U}, i \geq 2)$ are also independent. Now define random variables $(\Gamma_u)_{u\in\mathbb{U}}$ inductively by setting $\Gamma_\varnothing = 1$ and, for $v \in \mathbb{U}$,

- $\Gamma_{v*1} = \Gamma_v$; and
- for each $i, j \geq 2$, $\Gamma_{v*i} = \Gamma_v \left(\frac{3}{4}\right)^{j-1}$ if and only if $\tau_{v,j-1} + 1 < i \leq \tau_{v,j} + 1$.

We have $Q_{v*i} \leq Q_v \cdot (3/4)^{j-1}$ for all $i > \tau_{v,j-1} + 1$: indeed, $j - 2$ factors $3/4$ come from the fact that $V_{v*\tau_{v,k}} \leq 3/4$ for each $2 \leq k \leq j - 1$, and the last factor $3/4$ comes from the $1/2$ involved in the recursive expression of Q_{v*i}. By construction of Q and $\Gamma = (\Gamma_u)_{u\in\mathbb{U}}$, it then follows that $Q_u \leq \Gamma_u$ for all $u \in \mathbb{U}$, so $N_x(Q) \leq N_x(\Gamma)$.

Next, define a function $\chi : \mathbb{U} \to \mathbb{U}$ as follows. First, $\chi(\varnothing) = \varnothing$. Inductively, given that $\chi(v) = u$, set $\chi(v * 1) = u * 1$ and, for $i \geq 2$, set $\chi(v * i) = u * j$ if and only if $\tau_{v,j-1} + 1 < i \leq \tau_{v,j} + 1$. Equivalently, for all $v \in \mathbb{U}$, $\chi(v)$ is the unique node u for which $(\gamma(w))_{\varnothing\preceq w\preceq u} = (\Gamma_w)_{\varnothing\preceq w\preceq v}$ (see Figure 1 for an example). It follows that $\chi(v) \in E_x(\gamma)$ if and only if $v \in E_x(\Gamma)$, which means that

$$E_x(\Gamma) = \{v \in \mathbb{U} : \chi(v) \in E_x(\gamma)\}.$$

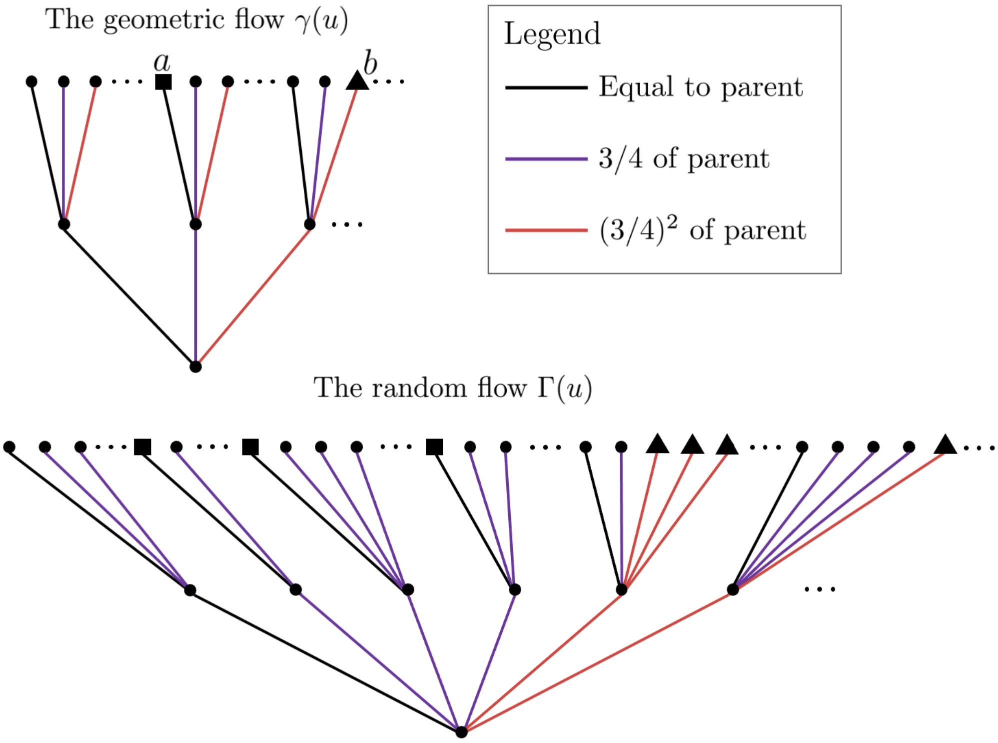

FIGURE 1. Illustration of the map χ and of the variables Z_u. The colour of each edge indicates the ratios of the values taken by the flow between the upper-end and the lower-end of the edge. Two vertices a and b are marked in the tree at the top. In the tree at the bottom, three nodes have a as their χ-image (marked with squares), meaning that $Z_a = 3$. Four nodes have b as their χ-image (marked with triangles), meaning that $Z_b = 4$.

So, writing $Z_u = |\chi^{-1}(u)|$ (again, see Figure 1 for an example), then

$$N_x(P) \le N_x(Q) \le N_x(\Gamma) = |E_x(\Gamma)| = \sum_{u \in E_x(\gamma)} Z_u. \tag{4.4}$$

Additionally, by the definition of χ, for all $v \in \mathbb{U}$ we have $Z_{v*1} = Z_v$, and for all $i \ge 2$,

$$Z_{v*i} = \sum_{u \in \chi^{-1}(v)} (\tau_{v,i} - \tau_{v,i-1}).$$

Using the independence of the increments $\tau_{v,i} - \tau_{v,i-1}$, it follows by induction that for all $u \in \mathbb{U}$, Z_u is distributed as the size $S_{\mathrm{r}(u)}$ of the $\mathrm{r}(u)$'th generation in a branching process with Geometric(1/2) offspring distribution, and thus (see Harris [14, Chapter I, Section 7.1]) that Z_u is itself Geometric($2^{-\mathrm{r}(u)}$)-distributed.

The above fact, together with the bound $\log(1-s) \le -s$, yields that

$$\mathbb{P}(Z_u \ge z) \le (1 - 2^{-\mathrm{r}(u)})^{z-1} \le 2\exp(-2^{-\mathrm{r}(u)} z)$$

for any $z \ge 1$. Lemma 4.6 provides that $\sup_{u \in E_x(\alpha)} \mathrm{r}(u) \le c_1 \sqrt{\log x}$ for all $u \in \mathbb{U}$, where $c_1 > 0$ is some constant; thus, for all $u \in \mathbb{U}$ and $z \ge 1$,

$$\mathbb{P}(Z_u \ge z 2^{c_1 \sqrt{\log x}}) \le \mathbb{P}(Z_u \ge z \cdot 2^{\mathrm{r}(u)}) \le 2e^{-z}. \tag{4.5}$$

Recalling that $N_x(\gamma) = |E_x(\gamma)|$, and using (4.5), (4.4) and a union bound, it follows that

$$\mathbb{P}\big(N_x(P) \ge N_x(\gamma) \cdot z \exp(c_1 \log(2) \sqrt{\log x})\big) \le 2N_x(\gamma)e^{-z}.$$

Finally, choose $z = y + \log N_x(\gamma) \le y N_x(\gamma)$. By Proposition 2.4 applied with $\alpha = 4/3$, we have $N_x(\gamma) \le \exp(c_2 + c_2\sqrt{\log_{4/3}(x)})$, where $c_2 > 0$ is another constant. This yields that

$$\begin{aligned}
&\mathbb{P}\Big(N_x(P) \ge y \exp\Big(2c_2 + 2c_2\sqrt{\log_{4/3} x} + c_1 \log(2)\sqrt{\log x}\Big)\Big) \\
&\le \mathbb{P}\big(N_x(P) \ge N_x(\gamma) \cdot z \exp(c_1 \log(2)\sqrt{\log x})\big) \\
&\le 2N_x(\gamma)e^{-z} \\
&= 2e^{-y},
\end{aligned}$$

which implies the desired inequality. □

4.3. Competitive ratio

The goal of this section is to prove the following analogue of Proposition 3.6 for the UA model.

Proposition 4.7. *For any $n \ge 1$, let $T_n \sim \mathrm{UA}(n)$. There exist universal constants $C, c > 0$ such that $\limsup_{n\to\infty} \mathbb{P}(\Phi(T_n) \ge x) \le Cx^{-c}$ for all $x \ge 1$.*

Before proceeding with the proof, recall from Section 4.1 that $P_u \overset{\text{a.s.}}{=} \lim_{n\to\infty} |\theta_u T_n|/n$ and that there exists a family $(U_u)_{u\in\mathbb{U}}$ of independent Uniform$[0,1]$-distributed random variables such that for all $u \in \mathbb{U}$ and $i \in \mathbb{N}_1$, $P_{u*i} \overset{\text{a.s.}}{=} P_u \cdot U_{u*i} \prod_{j=1}^{i-1}(1 - U_{u*j})$.

Proof. For $u \in \mathbb{U}$, let $j(u) = \arg\max(U_{u*i}\prod_{j=1}^{i-1}(1-U_{u*j}), i \ge 1)$, so that $P_{u*j(u)} = \max_{i\ge1} P_{u*i} = P_u \cdot U_{u*j(u)} \prod_{j=1}^{j(u)-1}(1-U_{u*j})$. The preceding maximum is almost surely achieved for a unique $i \ge 1$, so $j(u)$ is a.s. well-defined. Write $V_u = \max(U_{u*1}, 1-U_{u*1}) \sim$ Uniform$[1/2, 1]$ for $u \in \mathbb{U}$. Note that for all $u \in \mathbb{U}$ we have $U_{u*j(u)}\prod_{j=1}^{j(u)-1}(1-U_{u*j}) \le \max(U_{u*1}, 1-U_{u*1}) = V_u$. Thus, for all $u \in \mathbb{U}$, we have $P_u \le \prod_{v\prec u} V_v$.

Now define a random path $(u_i)_{i\ge0}$ in $\mathbb{U}$ as follows. Let $u_0 = \varnothing$ and, inductively, for $i \ge 1$, given u_i set $u_{i+1} = u_i * j(u_i)$. Since the families $((U_{u*i})_{i\ge1}), u \in \mathbb{U})$ are IID, the pairs $(V_u; j(u))_{u\in\mathbb{U}}$ are IID, which in turn implies that the random variables $(V_{u_i})_{i\ge0}$ are IID. Then let

$$K = \inf\{k \ge 1 : P_{u_k} \le 1/2\} \le \inf\left\{g \ge 1 : \prod_{i=1}^{g} V_{u_{j-1}} \le 1/2\right\} = G,$$

the final equality constituting the definition of G; it is straightforward to see that $G < \infty$ almost surely. Also, by (4.3) each P_u is a product of a finite number of independent uniform random variables so has a continuous distribution; thus, almost surely none of the P_u equal $1/2$, so in particular $P_{u_K} < 1/2$ almost surely.

Since $|\theta_u T_n|/n \xrightarrow{\text{a.s.}} P_u$ for all $u \in \mathbb{U}$ and K is a.s. finite, it follows that there exists a random variable $n_0 \in \mathbb{N}_1$ such that almost surely, for all $n \ge n_0$, $|\theta_{u_i} T_n| > n/2$ for all $i \in [K-1]$. We claim that by choosing n_0 large enough, we may also ensure that $|\theta_{u_{K-1}*j} T_n| < n/2$ for all $j \ge 1$. (This would be immediate from the convergence fact that $|\theta_u T_n|/n \xrightarrow{\text{a.s.}} P_u$ for all u, if we were only to consider finitely many values of j.) To see this, let J be the smallest integer $j \ge j(u_{K-1})$ for which $P_{u_{K-1}*1} + \ldots + P_{u_{K-1}*j} > P_{u_{K-1}}/2$. Then J is a.s. finite, so for n sufficiently large, $|\theta_{u_{K-1}*\ell} T_n| < n/2$ for $1 \le \ell \le J$. For such n we also have

$$\frac{|\theta_{u_{K-1}*1} T_n|}{n} + \ldots + \frac{|\theta_{u_{K-1}*J} T_n|}{n} \xrightarrow{\text{a.s.}} P_{u_{K-1}*1} + \ldots + P_{u_{K-1}*J} > \frac{P_{u_{K-1}}}{2},$$

so for n sufficiently large

$$\frac{1}{n}\sum_{\ell>J} |\theta_{u_{K-1}*\ell} T_n| = \frac{1}{n}\Big(|\theta_{u_{K-1}} T_n| - 1 - |\theta_{u_{K-1}*1} T_n| - \ldots - |\theta_{u_{K-1}*J} T_n|\Big) < \frac{P_{u_{K-1}}}{2} \le \frac{1}{2}.$$

Thus, for such n we also have $|\theta_{u_{K-1}*\ell}T_n| < n/2$ for $\ell > J$, as claimed.
It follows from the preceding paragraph that for $n \geq n_0$, $\varnothing = u_0 \preceq \cdots \preceq u_{K-1}$ is the sequence of nodes described in Lemma 2.3, so almost surely, for $n \geq n_0$,

$$\Phi(T_n) \leq \prod_{i=1}^{K-1} \frac{1}{1 - |\theta_{u_i}T_n|/n}.$$

Since $|\theta_{u_i}T_n|/n \xrightarrow{\text{a.s.}} P_{u_i}$, it follows that a.s. $\limsup_{n\to\infty} \Phi(T_n) \leq \prod_{i=1}^{K-1}(1 - P_{u_i})^{-1}$, so

$$\limsup_{n\to\infty} \mathbb{P}(\Phi(T_n) \geq x) \leq \mathbb{P}\left(\prod_{i=1}^{K-1} \frac{1}{1-P_{u_i}} \geq x\right) \leq \mathbb{P}\left(\prod_{i=1}^{G-1} \frac{1}{1 - V_{u_0} \cdot \ldots \cdot V_{u_{i-1}}} \geq x\right),$$

where for the last inequality we have used that $K \leq G$ and that $P_{u_i} \leq \prod_{j=1}^{i} V_{u_{j-1}}$.
Finally, we have

$$\mathbf{E}\big[(1 - V_1)^{-1/2}\big] = \int_0^1 \frac{\mathrm{d}x}{\sqrt{1 - \max(x, 1-x)}} = 2\int_0^{1/2} \frac{\mathrm{d}x}{\sqrt{x}} = 2\sqrt{2},$$

so the result follows by Lemma 3.7. □

4.4. Proof of the main theorem

Proof of Theorem 1.1. Recall that $|T_n| = n$: this will be used below to write some expressions more succinctly. Exactly like in the proof of Theorem 1.2, we partition the set $B_n = \{v \in T_n : \varphi_{T_n}(v) \leq \varphi_{T_n}(\varnothing)\}$ of all better candidates than the root into subtrees stemming from children of elements of $H_n = \{v \in T_n : |\theta_v T_n| \geq \frac{1}{3}n\}$. Namely, if $v \in B_n \setminus H_n$ then it has a unique ancestor $u \in T_n \setminus H_n$ whose parent is in H_n.
Here, in contrast to the proof of Theorem 1.2, the nodes in H_n can have an arbitrarily large number of children. To control the number of children that matter, i.e. those that have a descendant in B_n, we rely on Lemma 2.2. Indeed, for u an ancestor of $v \in B_n$, Lemma 2.2 gives us that $|\theta_v T_n|/n \geq (1 + \Phi(T_n))^{-1}$ so also $|\theta_u T_n|/n \geq (1 + \Phi(T_n))^{-1}$. Therefore, setting

$$M_n = \max_{u' \in H_n} \big|\{u \in T_n : \overleftarrow{u} = u', (1 + \Phi(T_n))|\theta_u T_n| \geq n\}\big|,$$

we obtain that

$$|B_n| \leq |H_n| + M_n \cdot |H_n| \cdot \max_{u \in T_n \setminus H_n} \mathbf{1}_{\{(1+\Phi(T_n))|\theta_u T_n| \geq n\}} \big|\{v \in \theta_u T_n : u * v \in B_n\}\big|. \tag{4.6}$$

Like for Theorem 1.2, we now want to control the sizes of the sets $\{v \in \theta_u T_n : u * v \in B_n\}$, for u appearing in the above maximum, by using Proposition 4.5. However, unlike the bound from Lemma 3.4, the bound provided by Proposition 4.5 only holds with high probability and only concerns the asymptotic proportions of nodes lying in subtrees of T_n. Hence, to avoid summing the probabilities of too many bad events or trying to prove rate of convergence bounds, we will basically show that B_n is contained with high probability in a deterministic finite set of words. To do so, we rely on Lemmas 2.2 and 4.2, and Proposition 4.7, to obtain that with high probability, all nodes u with $\varphi_{T_n}(u) \leq \varphi_{T_n}(\varnothing)$ will satisfy that $|\theta_u T_n|$ is large and $\mathrm{w}(u)$ is small. This will also be useful for bounding M_n and $|H_n|$. We now proceed to the details.
By Proposition 4.7, there exist constants $C, c_1 > 0$ such that for all $\varepsilon \in (0, 1)$,

$$\limsup_{n\to\infty} \mathbb{P}(\Phi(T_n) \geq \varepsilon^{-c_1}) \leq C\varepsilon. \tag{4.7}$$

Moreover, letting $c_2 = \frac{1+2c_1}{\log(3/2)}$, Lemma 4.2 yields that

$$\limsup_{n\to\infty} \mathbb{P}\big(\exists u \in \mathbb{U} : \mathrm{w}(u) \geq c_2 \log \tfrac{1}{\varepsilon}, |\theta_u T_n| \geq \tfrac{1}{2}\varepsilon^{c_1} n\big) \leq 6\varepsilon \tag{4.8}$$

for any $\varepsilon \in (0,1)$.

The above bound motivates us to define $\mathbb{U}^{(\varepsilon)} = \{v \in \mathbb{U} : \mathrm{w}(v) \le c_2 \log(1/\varepsilon)\}$; note that this set is deterministic and does not depend on n. Letting $E(n,\varepsilon)$ be the event that $\Phi(T_n) \le \varepsilon^{-c_1}$ and that $\{u \in T_n : |\theta_u T_n| \ge \frac{1}{2}\varepsilon^{c_1} n\} \subset \mathbb{U}^{(\varepsilon)}$, we have shown that $\liminf_{n\to\infty} \mathbb{P}(E(n,\varepsilon)) \ge 1-(C+6)\varepsilon$. We now prove several bounds assuming that $E(n,\varepsilon)$ occurs; in the below list we work on the event $E(n,\varepsilon)$.

(1) Since $(1+\Phi(T_n))^{-1} \ge (2\Phi(T_n))^{-1} \ge \frac{1}{2}\varepsilon^{c_1}$, if $u \in T_n$ has $|\theta_u T_n| \ge (1+\Phi(T_n))^{-1} n$ then $u \in \mathbb{U}^{(\varepsilon)}$.

(2) For $u' \in T_n$, the number of children u of u' with $|\theta_u T_n| \ge (1+\Phi(T_n))^{-1} n$ is bounded by the maximum of their last letters, which is itself bounded by the maximum of their weights. Hence, M_n is at most the maximum weight of a node $u \in \mathbb{U}^{(\varepsilon)}$, so $M_n \le c_2 \log(1/\varepsilon)$.

(3) Since H_n has at most 3 leaves, we have $|H_n| \le 3\max_{u\in H_n} \mathrm{ht}(u) \le 3\max_{u\in H_n} \mathrm{w}(u)$, and so if $\frac{1}{3} \ge \frac{1}{2}\varepsilon^{c_1}$ then $H_n \subset \mathbb{U}^{(\varepsilon)}$ and $|H_n| \le 3c_2 \log(1/\varepsilon)$.

(4) For any $u \in \mathbb{U}$, by Lemma 2.2, if $u \in B_n$ then $|\theta_u T_n| \ge (1+\Phi(T_n))^{-1} n \ge \frac{1}{2}\varepsilon^{c_1} n$, so $u \in \mathbb{U}^{(\varepsilon)}$. It follows that if $u * v \in B_n$ then $\mathrm{w}(v) \le \mathrm{w}(u*v) \le c_2 \log(1/\varepsilon)$.

(5) Finally, it holds that $\Phi(T_n)^{-1} \le \varphi_{T_n}(u)/\varphi_{T_n}(\varnothing)$ by definition, so if $u*v \in B_n$ then $\varphi_{T_n}(u)/\varphi_{T_n}(u*v) \ge \varepsilon^{c_1}$.

For $\varepsilon \in (0,1/3)$, on $E(n,\varepsilon)$, the above bounds and (4.6) entail that

$$|B_n| \le 3c_2 \log(\tfrac{1}{\varepsilon}) + 3c_2^2 \log^2(\tfrac{1}{\varepsilon}) \max_{u\in\mathbb{U}^{(\varepsilon)}, u\in T_n\setminus H_n} \left|\left\{ v \in \mathbb{U}^{(\varepsilon)} : \frac{\varphi_{T_n}(u)}{\varphi_{T_n}(u*v)} \ge \varepsilon^{c_1} \right\}\right|. \tag{4.9}$$

We are now almost ready to apply Proposition 4.5. Let $u \in \mathbb{U}$. For all $v \in \mathbb{U}$, we define $P_v^{(u)} = P_{u*v}/P_u \overset{\text{a.s.}}{=} \lim_{n\to\infty} |\theta_{u*v} T_n|/|\theta_u T_n|$. It then follows from (2.4) that if $P_u \le \frac{1}{3}$ then almost surely

$$\lim_{n\to\infty} \frac{\varphi_{T_n}(u)}{\varphi_{T_n}(u*v)} = \prod_{\varnothing \prec w \preceq v} \frac{P_{u*w}}{1-P_{u*w}} \le \prod_{\varnothing \prec w \preceq v} \frac{P_w^{(u)}}{2}$$

for all $v \in \mathbb{U}$, and so in particular for all $v \in \mathbb{U}^{(\varepsilon)}$. Also recall that $|\theta_u T_n|/n$ almost surely converges to P_u, which yields that $\limsup_{n\to\infty} \mathbf{1}_{\{u\in T_n\setminus H_n\}} \le \mathbf{1}_{\{P_u \le 1/3\}}$ almost surely. Since $\mathbb{U}^{(\varepsilon)}$ is deterministic and finite, it follows from the above bounds that almost surely

$$\begin{aligned}
\limsup_{n\to\infty} \mathbf{1}_{\{u\in T_n\setminus H_n\}} \left|\left\{ v \in \mathbb{U}^{(\varepsilon)} : \frac{\varphi_{T_n}(u)}{\varphi_{T_n}(u*v)} \ge \varepsilon^{c_1} \right\}\right| &\le \left|\left\{ v \in \mathbb{U}^{(\varepsilon)} : \prod_{\varnothing \prec w \preceq v} \frac{P_w^{(u)}}{2} \ge \varepsilon^{c_1} \right\}\right| \\
&\le \left|\left\{ v \in \mathbb{U} : \prod_{\varnothing \prec w \preceq v} \frac{P_w^{(u)}}{2} \ge \varepsilon^{c_1} \right\}\right| \\
&\overset{d}{=} N_{\varepsilon^{-c_1}}(P)\,,
\end{aligned}$$

where the final identity in distribution follows from (4.3) and Proposition 4.1. To obtain a tail bound on $N_{\varepsilon^{-c_1}}(P)$, we apply Proposition 4.5 with $y = (\log(2)c_2 + 1)\log(1/\varepsilon)$; this yields

$$\mathbb{P}\Big(N_{\varepsilon^{-c_1}}(P) \ge y \exp\big(c + c\sqrt{\log(\varepsilon^{-c_1})}\big)\Big) \le 2\exp(-y) = 2\varepsilon^{c_2 \log 2 + 1}.$$

Since $s \le \exp(\sqrt{s})$ for all $s > 0$, there exists $c_3 > 0$ such that $y\exp\big(c + c\sqrt{\log(\varepsilon^{-c_1})}\big) \le \exp(c_3 + c_3\sqrt{\log\frac{1}{\varepsilon}})$, which together with the two preceding displays yields that

$$\limsup_{n\to\infty} \mathbb{P}\bigg(u \in T_n\setminus H_n\,;\, \left|\left\{ v\in\mathbb{U}^{(\varepsilon)} : \frac{\varphi_{T_n}(u)}{\varphi_{T_n}(u*v)} \ge \varepsilon^{c_1} \right\}\right| \ge \exp\big(c_3 + c_3\sqrt{\log\tfrac{1}{\varepsilon}}\big)\bigg) \le 2^{-c_2 \log 1/\varepsilon} 2\varepsilon.$$

Since $|\mathbb{U}_\varepsilon| \leq 2^{c_2 \log 1/\varepsilon}$, a union bound then gives

$$\limsup_{n\to\infty} \mathbb{P}\left(\max_{u\in\mathbb{U}^{(\varepsilon)}} \mathbf{1}_{\{u\in T_n\setminus H_n\}} \left| \left\{ v\in \mathbb{U}^{(\varepsilon)} : \frac{\varphi_{T_n}(u)}{\varphi_{T_n}(u*v)} \geq \varepsilon^{c_1} \right\} \right| \geq \exp\left(c_3 + c_3\sqrt{\log \tfrac{1}{\varepsilon}}\right) \right) \leq 2\varepsilon. \tag{4.10}$$

Finally, we combine (4.9) and (4.10) with the fact that $\limsup_{n\to\infty} \mathbb{P}(E(n,\varepsilon)^c) \leq (C+6)\varepsilon$ to deduce that for all $\varepsilon \in (0, 1/3)$,

$$\limsup_{n\to\infty} \mathbb{P}\Big(|B_n| \geq 3c_2\log(\tfrac{1}{\varepsilon}) + 3c_2^2 \log(\tfrac{1}{\varepsilon})^2 \exp\left(c_3 + c_3\sqrt{\log\tfrac{1}{\varepsilon}}\right)\Big) \leq (8+C)\varepsilon.$$

Since we always find the root when $K \geq |B_n|$, the desired result follows. □

4.5. Reducing the d-dependence in Theorem 1.2.

Let $d \geq 2$. Here, we use the notation of Section 3.1, so that T_n stands for a random plane tree with $T_n \sim \mathrm{UA}_{d+1}(n)$. In particular, recall the random variables $(D_u)_{u\in\mathbb{U}\setminus\{\varnothing\}}$ and that for $u = (u_1, \ldots, u_h)$,

$$P_u = \prod_{i=1}^{h} D_{(u_1,\ldots,u_i)}.$$

As already explained in Remark 3.4, the only step of the proof of Theorem 1.2 that introduced a dependence on d in the constant c_d^* is the deterministic control of the number of competitors in small subtrees given by Lemma 3.4 and Proposition 3.5. To remove this dependence, we can instead rely on a probabilistic analysis of the UA_{d+1} model and follow the strategy used for studying the general uniform attachment model more closely. Indeed, one can check that by rerunning the proof of Theorem 1.1, but using Proposition 3.1 in place of Proposition 4.1 and replacing $\mathbb{U}^{(\varepsilon)} = \{u \in \mathbb{U} : \mathrm{w}(u) \leq c_2 \log 1/\varepsilon\}$ with the finite set $\{u \in \mathbb{U}_d : \mathrm{ht}(u) \leq c_2 \log 1/\varepsilon\}$, we obtain another proof of Theorem 1.2 with a constant c^* that does not depend on d, provided we establish the following analogue of Proposition 4.5.

Proposition 4.8. *Let us set $P' = (P_{1*u}/P_1)_{u\in\mathbb{U}}$. Then there exists a universal constant $c > 0$, that does not depend on d, such that for any $x, y \geq 1$,*

$$\mathbb{P}\big(N_x(P') \geq y\exp(c + c\sqrt{\log x})\big) \leq 2e^{-y}.$$

The reason we need P' rather than P in this proposition is that in the $(d+1)$-regular model, the split at the root is different from that of all other nodes in $\mathbb{U}_d$; we could equally have defined $P' = (P_{v*u}/P_v)_{u\in\mathbb{U}}$ for any $v \in \mathbb{U}_d \setminus \{\varnothing\}$.

Next, carefully revisiting the proofs of Proposition 4.5, Lemma 4.6, and Corollary 4.4, one can verify that Proposition 4.8 would follow from the adaptation of Lemma 4.3 stated below (in which, informally, the change from the UA to the UA_{d+1} model forces us to replace the values $4/3$ and $1/2 = \mathbb{P}(V_1 \leq 3/4)$ with other universal constants $\alpha > 1$ and $p = \mathbb{P}(W_1 \leq \alpha^{-1}) > 0$).

Lemma 4.9. *There exists a probability measure μ on $[0, 1]$ with $\mu(\{1\}) < 1$ such that the following holds for all $d \geq 2$. If $(D_1, \ldots, D_d) \sim \mathrm{Dir}_d(\frac{1}{d-1})$, then there is a sequence $(W_i)_{i\geq1}$ of* IID *random variables with distribution μ and a random permutation σ of $[d]$ such that almost surely $D_{\sigma(i)} \leq \frac{1}{2}\prod_{j=1}^{i-2} W_j$ for all $2 \leq i \leq d$.*

We now explain how to prove Lemmma 4.9. We seek an admissible probability measure μ of the form $p\delta_{\alpha^{-1}} + (1-p)\delta_1$ with $\alpha > 1$ and $p > 0$. Our argument is based on a sampling method for the symmetric Dirichlet distribution via *stick-breaking* which builds the components of the random vector in an expression similar to $U_i \cdot \prod_{j=1}^{i-1}(1-U_j)$. This will put us in a good position to use the proof method of Lemma 4.3.

Let π be a uniformly random permutation of $[d]$. Independently from π, let $X_1, \dots, X_{d-1}$ be independent random variables such that $X_i \sim \mathrm{Beta}(\frac{d}{d-1}, 1 - \frac{i-1}{d-1})$ for all $i \in [d-1]$. Then, set $Y_d = \prod_{j=1}^{d-1}(1 - X_j)$ and $Y_i = X_i \cdot \prod_{j=1}^{i-1}(1 - X_j)$ for all $i \in [d-1]$. It holds that

$$(Y_{\pi(1)}, \dots, Y_{\pi(d)}) \sim \mathrm{Dir}_d\big(\tfrac{1}{d-1}\big)\,. \tag{4.11}$$

This identity in distribution can be derived from the fact that for any integers $j_1, \dots, j_k \in \mathbb{N}_1$, $\mathrm{Dir}_k(\frac{j_1}{d-1}, \dots, \frac{j_k}{d-1})$ is the limit law of the proportion-vector in a k-colour Pólya urn starting with j_i balls of the i'th colour for all $i \in [k]$, and where each ball drawn is returned along with $d-1$ new balls of the same colour; see e.g. [2] or [8, Section 6]. Then (4.11) describes the case where $j_1 = \ldots = j_{d-1} = 1$: the permutation π^{-1} represents the order in which the colours are drawn for the first time and for $i \in [d-1]$, X_i is the relative proportion of colour $\pi^{-1}(i)$ with respect to the colours that have yet to be drawn.

Thanks to (4.11), we only need to construct IID random variables $W_1, \dots, W_{d-2}$ with common distribution $\mu = p\delta_{\alpha^{-1}} + (1-p)\delta_1$ and a random permutation σ of $[d]$ such that $Y_{\sigma(i)} \le \frac{1}{2}\prod_{j=1}^{i-2} W_j$ for all $2 \le i \le d$. Similarly to as in the proof of Lemma 4.3, we consider $K = \min\{d\} \cup \{i \in [d-1] : X_i > \frac{1}{2}\}$ and define $\sigma(i) = K\mathbf{1}_{\{i=1\}} + (i-1)\mathbf{1}_{\{2 \le i \le K\}} + i\mathbf{1}_{\{i \ge K+1\}}$ for all $i \in [d]$. Then, to be able to obtain the same deterministic inequalities as in the proof of Lemma 4.3, we must construct the $W_1, \dots, W_{d-2}$ so that it holds that

$$W_i \ge \begin{cases} 1 - X_i & \text{if } i \le K-1, \\ 1 - X_{i+1} & \text{if } i \ge K, \end{cases}$$

almost surely for all $i \in [d-2]$ while ensuring their independence. By independence of $X_1, \dots, X_{d-1}$, this is possible as soon as for all $i \in [d-2]$,

$$\mathbb{P}(1 - X_i > \alpha^{-1} \,|\, X_i \le 1/2) \le 1 - p \quad \text{and} \quad \mathbb{P}(1 - X_{i+1} > \alpha^{-1}) \le 1 - p.$$

It follows from the above discussion that Lemma 4.9 — and, hence, a version of Theorem 1.2 with the constant c_d^* replaced by a constant $c^* > 0$ which is independent of d — is a consequence of the following uniform estimate for Beta distributions.

$$\forall a \in [1,2], \forall b \in (0,1], \quad \text{if} \quad X \sim \mathrm{Beta}(a,b) \quad \text{then} \quad \mathbb{P}(X < 1/17 \,|\, X \le 1/2) \le \frac{1}{2}. \tag{4.12}$$

More concretely, this bound yields that Lemma 4.9 holds with $\mu = \frac{1}{2}\delta_{16/17} + \frac{1}{2}\delta_1$.

Proof of (4.12). Let B, B' be two random variables such that $B' \sim \mathrm{Beta}(1,b)$ and $B \sim \mathrm{Beta}(2,b)$. From (2.12), we have the stochastic dominations $B' \preceq_{\mathrm{st}} X \preceq_{\mathrm{st}} B$, and so

$$\mathbb{P}(X < 1/17 \,|\, X \le 1/2) \le \frac{\mathbb{P}(B' < 1/17)}{\mathbb{P}(B \le 1/2)}\,.$$

We bound the numerator and the denominator separately. Since $\Gamma(1+b) = b\Gamma(b)\Gamma(1)$,

$$\mathbb{P}(B' < 1/17) = b\int_0^{1/17} (1-x)^{b-1}\,\mathrm{d}x \le b\int_0^{1/17} (1-x)^{-1}\,\mathrm{d}x \le \frac{b}{17}(16/17)^{-1} = \frac{b}{16}$$

because $0 < b \le 1$. Similarly, since $\Gamma(2+b) = (1+b)\Gamma(1+b) = (1+b)b\Gamma(b)\Gamma(2)$,

$$\mathbb{P}(B \le 1/2) = (1+b)b\int_0^{1/2} x(1-x)^{b-1}\,\mathrm{d}x \ge b\int_0^{1/2} x\,\mathrm{d}x = \frac{b}{8},$$

which concludes the proof. ∎

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