Three branching anecdotes
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Dynamics on Random Graphs and Random Maps
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THE CONFERENCE MORNING SESSION

Welcome, everyone!

Sorry, I haven't had my coffee yet...

(Awkward silence)

Thanks for attending. I couldn't find an earlier flight.

DAY 1
7:00am

DAY 2
7:00am

DAY 3
7:00am

LAST DAY
7:00am

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Most trees are short and fat.
Trees:

\( T \)

- \( h^+(T) = 3 \)
- \( \text{wid}(T) = 4 \)

**Height**: Greatest distance from any node to the root

\( h^+(T) \)

**Width**: Greatest \# nodes on a single level.

\( \text{wid}(T) \)
Main Results Fix any r.v. $C$ with $\sum_{k \geq 0} P(C=k) = 1$, write $p_k = P(C=k)$. Let $T$ be $\text{GW}(C)$ distributed.

**Theorem** ("Most trees are short & fat") There is a universal constant $\delta > 0$ s.t.

$$P(\text{ht}(T) \geq \frac{k}{1-p_1} \cdot \text{wid}(T)) \leq \exp(-\delta k).$$

**Remark:** If $E[C] > 1$ then $P(\sigma = \infty) > 0$, and

$$P(\text{ht}(T) = \text{wid}(T) = \infty | \sigma = \infty) = 1.$$

Also, given that $\sigma < \infty$, the cond. dist. of $T$ is $\text{GW}(\hat{C})$ where $P(\hat{C} = 1) = p_1$,

$$E(\hat{C}) < 1$$

so can assume $E[C] \leq 1$.

**Heuristic:** $\text{GW}$ trees satisfy $\text{wid}(T) \cdot \text{ht}(T) \approx \text{vol}(T) = \sigma$

Implies "$\text{ht} > C^2 \cdot \text{wid}" \approx "\text{ht}^2 \approx C^2 \cdot \text{vol}"$ so

$$P(\text{ht}(T) > \frac{k}{\sqrt{1-p_1}} \sqrt{\text{vol}(T)}) \leq \exp(-\delta k^2)$$

**Theorem:** $P(\text{ht}(T) > \frac{k}{\sqrt{1-p_1}} \sqrt{\text{vol}(T)}) \leq \exp(-\delta k^2)$
Galton-Watson Trees

- Each node has random # of children
- Nodes reproduce independently

Construction

- \((C_i, i \geq 1)\) independent copies of a random variable \(C\) with \(\sum_{k \geq 0} P(C = k) = 1\).

Rule

The sequence \((C_i, i \geq 1)\) gives # children of nodes, in breadth-first search order.

Example: \((2, 1, 3, 0, 0, 1, 0, 2, 0, 0, 1, 4, 0, \ldots)\)

Halting Condition

Trees with \(n\) nodes have \(n-1\) edges.

For \(n \geq 0\)

# nodes discovered by time \(n\)

\[ = 1 + \sum_{i=1}^{n} C_i \]

# vertices explored by time \(n\)

\[ = n \]

Total # vertices = \(\sigma\)

\[ = \inf\{ n : 1 + \sum_{i=1}^{n} C_i = n \} \]
Setup
\[ \sum_{j=1}^{i} C_j = \# \text{nodes discovered by time } i. \]

Let \( S_i = 1 + \sum_{j=1}^{i} (C_j - 1) \),

\[ = \# \text{nodes in "BFS queue" at time } i \]

\[ EC \leq 1 \Rightarrow E S_n = 1 + n(IEC - 1) \leq 1. \]

\( \sigma = \inf \{ t : S_t = 0 \} \) = first time no nodes left to explore

**Prop:** Let \( W(T) = \max(S_i, 0 \leq i < \sigma) \).

Then \( \text{wid}(T) \in (W(T)/2, W(T)] \)

**Proof:** During BFS on level \( k \), "exploration queue" \( \subset T_k \cup T_{k+1} \)

and \( = T_k \) at start of level \( k \).

**Idea:**
\[ \text{ht}(T) = \frac{\sum_{k=1}^{\infty} 1}{T_k} = \sum_{k=1}^{\infty} \frac{A}{|T_k|}. \]

When \( v_i \in T_k \) then \( S_i \approx |T_k| \) so perhaps

\[ \text{ht}(T) \approx \sum_{k=1}^{\infty} \sum_{v_i \in T_k} \frac{1}{S_i} \approx \sum_{i=1}^{\sigma} \frac{1}{S_i} = H(T)? \]

[False; consider a star with \( n \) leaves. But...]

**Prop:**
\[ \text{ht}(T) \leq 3H(T). \]

**Corollary** Suffices to prove
\[ P(H(T) \geq \frac{k}{1-p} W(T)) \leq e^{-\delta k}, \]

thm. follows.
\[ W(\sigma) = \max(S_i, 0 \leq i < \sigma) \quad H(\sigma) = \sum_{i=1}^{\sigma} \frac{1}{S_i} \quad \text{Aim:} \quad \mathbb{P}(H(\sigma) \geq \frac{k}{1-p}, W(\sigma) \leq e^{-c} \text{ for all } \sigma) \]

**Key Tool:** Decomposition into scales.

When \( S_i \approx 2^l \) ("scale 2^l") for \( j \in \{i, \ldots, i+2^l\} \), have

\[ H(i+2^l) - H(i) = \sum_{j=i+1}^{i+2^l} \frac{1}{S_j} \approx 2^l \cdot \frac{1}{2^l} = 1. \]

So bound (a) time to change scales,

(b) "\# visits to scales" \( = (M(l), l \geq 1) \)

(a) **Thm** (Lévy; Doeblin; Kolmogorov; Rogozin; Le Cam; Esséen; Kesten):

With \( p = \max p_i \), have \( \max_k \mathbb{P}(S_n = k) \leq \frac{A p}{\sqrt{n(1-p)}} \) \( A > 0 \) universal.

"Any random walk spreads out over \( \geq \sqrt{n} \) values by time \( n \)." Here \( \sqrt{n} \approx 2^l \).

So leave scale \( 2^l \) after time \( \tau = O(p(4^l)) \).

Contribution to \( H \leq C \frac{\tau}{2^l} = O_p(2^l) \)

(b) **Fact:** Given that \( M(l) \neq 0 \), \( M(l) \) dominated by sum of 2 \( \text{Geom}(\frac{1}{2}) \) r.v.s; \( \Rightarrow \mathbb{P}(M(l) > k \mid M(l) > 0) \approx 2^{-k/2} \).

Proof via upcrossings. \( \blacksquare \)

With \( l_{\text{max}} = \max(l : M(l) \neq 0) \) get \( W \approx 2^{l_{\text{max}}} \), \( H = O_p(1) \cdot \sum_{l \leq l_{\text{max}}} 2^l M(l) = O_p(2^{l_{\text{max}}}) \)
Remarks

- Stronger results if add info. about tails of degrees.

**Ex:**

- If \( \mathbb{P}(C \geq k) = \Theta(t^{-\alpha}), \, \alpha \in (1,2), \) then \( \mathbb{P}(\text{ht}(T) > A \cdot m \cdot \text{wid}(T)^{\alpha-1}) \leq 2^{-\delta m} \)

- If \( \text{Var}(C) = \infty \) then \( \forall \varepsilon > 0 \exists n_0 \text{ s.t. for } x > 0, \quad n > x^2 n_0, \) \( \mathbb{P}(\text{ht}(T) > A \cdot m \cdot \text{wid}(T), |T| > n) \leq \frac{x}{n^{1/2}} e^{-x/\varepsilon} \)

- **Conjecture:** All this works even conditional on size of tree: \( \mathbb{P}(\text{ht}(T) > A \cdot m \cdot \text{wid}(T) | \sigma = n) \leq 2^{-\delta m} \)

- **Conjecture:** Binary trees are the tallest.

Consider random trees \( T_{\vec{n}} \) with a fixed degree seq \( \vec{n} = (n_i, i \geq 1) \).

Here \( n_i = \# \text{ nodes of deg } i \).

\( \vec{n} \) the number of children

To stochastically maximize \( \text{ht}(T_{\vec{n}}) \) among sequences with \( n_0 = n, \quad n_i = 0, \) choose the seq. \( (n, 0, n-1, 0, \ldots) \)
Comparing arbitrary trees to binary trees
Conjecture: Binary trees are the tallest.

Consider random trees $T_{\vec{n}}$ with a fixed degree seq $\vec{n}=(n_i, i \geq 1)$.

Here $n_i = \#$ nodes of deg $i$.

$\vec{n}$ = # children

To stochastically maximize $ht(T_{\vec{n}})$ among sequences with $n_0=n$, $n_i=0$,

choose the seq. $\text{bin}(n) = (n,0,n-1,0,...)$

"Evidence."

Prop: Let $\vec{n}$ have $n_0=n$, $n_i=0$.

Let $(T_{\vec{n}}, V)$ be random marked tree with deg. sequence $\vec{n}$

Let $(T_{\text{bin}(n)}, W)$ be random marked binary tree with $n$ leaves.

Then $height(V) \leq \text{st height}(W)$.
Warm-up:

- Number of binary trees, $n$ leaves:
  $$\frac{1}{2n-1} \binom{2n-1}{n}$$

- Number of trees with degree sequence $\vec{n}$. With $|\vec{n}| = \sum n_i$ then get:
  $$\frac{1}{|\vec{n}|!} \binom{|\vec{n}|}{n_i,i\geq 0} = \frac{1}{|\vec{n}|!} \frac{|\vec{n}|!}{n_i!}$$
  $$= \frac{1}{|\vec{n}|!} \cdot \# \text{ lattice walks, } n_i \text{ steps of size } i-1,$$

- Number of binary forests, $n$ leaves, $k$ connected components:
  $$\frac{k}{2n-k} \binom{2n-k}{n}$$

- Number of forests with degree sequence $\vec{n}$. With $k$ trees, then get:
  $$\frac{k}{|\vec{n}|!} \binom{|\vec{n}|}{n_i,i\geq 0}$$
  $$= \frac{k}{|\vec{n}|!} \cdot \# \text{ lattice walks, } n_i \text{ steps of size } i-1,$$

- Number of marked forests with degree sequence $\vec{n}$, mark in last tree:
  $$= |\vec{n}| \cdot \frac{1}{k} \cdot \# \text{ forests with degree seq. } \vec{n}$$
  $$= \binom{|\vec{n}|}{n_i,i\geq 0} = \# \text{ lattice walks, } n_i \text{ steps of size } i-1$$
Trunks of trees

Let \((T,W)\) be a random marked binary tree, \(n\) leaves.

**Trunk** = path from root to marked vertex, together with children of path vertices

\[ S_k \]

Marked binary trees with \(n\) leaves, trunk containing \(S_k\),
(marked node in subtree rooted at \(v_k\))

\[ = \text{# binary forests with } n \text{ leaves, } k \text{ trees, mark in last tree } = \binom{2n-k}{n} \]

Marked binary trees with \(n\) leaves, trunk containing \(S_{k+1}\),

\[ = \binom{2n-k-1}{n} \]

Ratio is

\[ \frac{(2n-k-1)! \cdot n! \cdot (n-k)!}{n!(n-k-1)! \cdot (2n-k)!} = \frac{n-k}{2n-k} \]

Two possible choices for \(v_{k+1}\) (left or right)

So \( P(W = v_k \mid \text{Spine contains } S_k) = 1 - 2 \cdot \frac{n-k}{2n-k} = \frac{k}{2n-k} \)
Trunks of trees

Let \((T, V)\) be a random marked tree with degree sequence \(\vec{n}\). Let \(\hat{\vec{m}} = (m_0, m_1, m_2, \ldots)\) be obtained from \(\vec{n}\) by removing degrees of \(v_1, \ldots, v_{k-1}\): 

\[
\text{Then } \# \text{ is } \binom{|\hat{\vec{m}}|-1}{\hat{\vec{m}}(d)}.
\]

Ratio is 

\[
\frac{\binom{|\hat{\vec{m}}|-1}{\hat{\vec{m}}(d)}}{\binom{|\vec{m}|}{\vec{m}(d)}} = \frac{\hat{m}_d}{|\hat{\vec{m}}|}.
\]

There are \(d\) possible choices for \(v_{k+1}\) (which child) 

\[
P(V\uparrow v_k, \deg(v_k) = d|\text{Trunk contains } S_k) = \frac{d \hat{m}_d}{|\hat{\vec{m}}|}
\]

So 

\[
P(V = v_k|\text{Trunk contains } S_k) = 1 - \frac{\sum d \hat{m}_d}{|\hat{\vec{m}}|} = \frac{1 + \sum (\deg(v_i) - 1)}{|\vec{m}| - (k-1)} \geq \frac{k}{2(n-k)}
\]
Prop: Let $\tilde{n}$ have $n_0 = n$, $n_1 = 0$. Let $(T_{\tilde{n}}, V)$ be random marked tree with deg. sequence $\tilde{n}$.

Let $(T_{\text{bin}(n)}, W)$ be random marked binary tree with $n$ leaves.

Then $\text{height}(V) \leq \text{height}(W)$.

Proof:

In marked binary tree $\Pr(W = v_k | \text{Trunk contains } S_k) = \frac{k}{2n-k}$.

In marked tree with degree sequence $\tilde{n}$, $\Pr(V = v_k | \text{Trunk contains } S_k) \geq \frac{k}{2n-k}$.

So can couple so that $W \leq V$. 

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This construction of a randomly sampled node seems useful.

Conjecture 21.5. If \( \nu = 1 \) and \( \sigma^2 = \infty \), then \( H(T_n)/\sqrt{n} \xrightarrow{P} 0 \).

Conjecture 21.6. If \( \nu = 1 \) and \( \sigma^2 = \infty \), then \( W(T_n)/\sqrt{n} \xrightarrow{P} \infty \).

Problem 21.7. Does \( \nu < 1 \) imply that \( H(T_n)/\sqrt{n} \xrightarrow{P} 0 \)?

Problem 21.8. Does \( \nu < 1 \) imply that \( W(T_n)/\sqrt{n} \xrightarrow{P} \infty \)? \( \checkmark \) (I think)

Furthermore, still in the case \( \nu \geq 1, \sigma^2 < \infty \), Addario-Berry, Devroye and Janson [1] have shown sub-Gaussian tail estimates for the height and width

\[
\mathbb{P}(H(T_n) \geq x\sqrt{n}) \leq Ce^{-cx^2}, \quad (21.12)
\]

\[
\mathbb{P}(W(T_n) \geq x\sqrt{n}) \leq Ce^{-cx^2}, \quad (21.13)
\]

uniformly in all \( x \geq 0 \) and \( n \geq 1 \) (with some positive constants \( C \) and \( c \) depending on \( \pi \) and thus on \( \mathbf{w} \)). In view of (21.11), we cannot expect (21.13) to hold when \( \sigma^2 = \infty \) (or when \( \nu < 1 \)), but we see no reason why (21.12) cannot hold; (21.10) suggests that \( H(T_n) \) typically is smaller when \( \sigma^2 = \infty \).

Problem 21.9. Does (21.12) hold for any weight sequence \( \mathbf{w} \) (with \( C \) and \( c \) depending on \( \mathbf{w} \), but not on \( x \) or \( n \))?

It follows from (21.10)–(21.11) and (21.12)–(21.13) that \( \mathbb{E}H(T_n)/\sqrt{n} \) and \( \mathbb{E}W(T_n)/\sqrt{n} \) converge to positive numbers. (In fact, the limits are \( \sqrt{2\pi}/\sigma \) and \( \sqrt{\pi}/2\sigma \), see e.g. Janson [61], where also joint moments are computed.)

Problem 21.10. What are the growth rates of \( \mathbb{E}H(T_n) \) and \( \mathbb{E}W(T_n) \) when \( \sigma^2 = \infty \) or \( \nu < 1 \)?

Can we use it to prove binary trees are tallest??
The MST of random 3-regular graphs
The MST problem

\( G = (V, E) \) finite connected graph.

A spanning forest is a subgraph \( F = (V, E') \), \( E' \subseteq E \), with no cycles. It is a spanning tree if it is also connected.

For fixed positive weights \( (U_e)_{e \in E} \), the weight of a spanning tree \( T \) of \( G \) is

\[
\omega(T) = \sum_{e \in E(T)} U_e
\]

Say \( T \) is the MST of \( G \) if

\[
\omega(T) = \min\{\omega(T'): T' \text{ a spanning tree of } G\}
\]

If weights \( U_e \) are all distinct then MST is unique.
Old thm (A-B, Broutin, Goldschmidt, Miermont 2017):
Give $K_n$ IID exchangeable edge weights, then
$n^{-\frac{1}{3}} \cdot \text{MST}(K_n) \xrightarrow{d} \mathcal{T}
\quad \mathcal{T} = \text{same random IR-tree.}$

New thm
Random 3-regular graph, $n$ vertices.
Give $G(n,3)$ IID exchangeable edge weights, then
$(6n)^{-\frac{1}{3}} \cdot \text{MST}(K_n) \xrightarrow{d} \mathcal{T}
\quad \mathcal{T} = \text{same random IR-tree.}$
Cycle breaking
Alg. for computing MST of \( G = (V,E) \)

- Let \( G_0 = G \).
- Order edges as \( e(1), \ldots, e(m) \) so \( \mathbb{U}(e(1)) > \cdots > \mathbb{U}(e(m)) \)
  For \( 0 \leq i < m \)
  - If \( G_i - e(i+1) \) is connected set \( G_{i+1} = G_i - e(i+1) \)
  - Else set \( G_{i+1} = G_i \).

NB: If edge weights are exchangeable then \( e(i+1) \)
is a u.rand. edge of \( G_i \).

One way to remove random edges: give them i.i.d. lengths,
run a Poisson process on \( G \) according to length measure,
cut where points fall.
Key Idea:

1. A-BBGM showed $T$ is well-approximated by $T^\lambda = \text{MST of largest component of } G(n, P_\lambda)$
   \[ P_\lambda = \frac{1}{n} + \frac{\lambda}{n^{4/3}} = "\lambda\text{-barely supercritical}" \]
   when $\lambda$ is large.

2. $T^\lambda \approx T^\lambda_{\text{CORE}} := \text{MST of CORE of largest component of } G(n, P_\lambda)$
   CORE = Graph remaining after removing all pendant subtrees.

3. CORE($\lambda$) \( \approx \) Random $3$-regular graph with $27^{3/2}$ vertices, exponential edge lengths
   [Can use cycle breaking via Poisson cuts to build MST of comp.; this is done in A-BBGM]

4. MST without edge lengths and with lengths related by a scaling factor
Universality, extensions

The tree $T$ is expected to be the MST scaling limit on any "high-dimensional" graph, e.g.

- random $d$-regular graphs
- the hypercube
- the lattice torus $(\mathbb{Z}/n\mathbb{Z})^d$ for $d > 8$

For low dimensions the picture is blurry.

$d = 2$: The MST scaling limit (as an embedded object) exists and has dimension $\epsilon \in (1+\epsilon, 7/4)$. (Garban, Pete, Schramm 20??)

Numerics suggest dimension $1.22...$ no non-numerical predictions in physics literature.

$3 \leq d \leq 6$, Wide open. $6 \leq d \leq 8$ Opinions vary.