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Vertex colouring edge partitions

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Abstract

A partition of the edges of a graph G into sets $\{S_1, \ldots, S_k\}$ defines a multiset X_v for each vertex v where the multiplicity of i in X_v is the number of edges incident to v in S_i. We show that the edges of every graph can be partitioned into 4 sets such that the resultant multisets give a vertex colouring of G. In other words, for every edge (u, v) of G, $X_u \neq X_v$. Furthermore, if G has minimum degree at least 1000, then there is a partition of $E(G)$ into 3 sets such that the corresponding multisets yield a vertex colouring.

Keywords: Edge weights; Vertex colours; Degree-constrained subgraphs

1. Introduction

A k-edge partition of a graph G is an assignment of an integer label between 1 and k to each of its edges. The edge partition is *proper* if no two incident edges receive the same label. It is *vertex distinguishing* if for every two vertices u and v, the multiset X_u of labels appearing on edges incident to u is distinct from the multiset X_v of labels appearing on edges incident to v. It is *vertex colouring* if for every edge (u, v), X_u is distinct from X_v. Proper vertex colouring edge partitions and proper vertex distinguishing edge partitions have been studied by many researchers [2–4] and are reminiscent of harmonious colourings (see [5]).

Clearly a graph cannot have a vertex colouring edge partition if it has a component which is isomorphic to K_2. We call a graph without such a component *nice*.

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In [7], Karoński et al. initiated the study of vertex colouring edge partitions. They proved that every nice graph permits a vertex colouring 183-edge partition and that graphs of minimum degree at least 10^{99} permit a vertex colouring 30-edge partition. In fact, they conjectured that for every nice graph, the edges can be labelled from $\{1, 2, 3\}$ and the vertices coloured by the *sum* of the incident edge labels (not just the multi-set). To see that two labels are not in general sufficient for either variant of the problem, consider K_3, or C_6 for a bipartite example. In this paper we show:

Theorem 1.1. *Every nice graph permits a vertex colouring* 4*-edge partition.*

Theorem 1.2. *Every nice graph of minimum degree* 1000 *permits a vertex colouring* 3*-edge partition.*

We prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3. In doing so, the following result from [7] will be useful:

Theorem 1.3. *Let M be an abelian group with $2k+1$ elements $\{m_1, \ldots, m_{2k+1}\}$, and let G be a nice, $2k+1$-colourable graph. Then there is a vertex colouring $2k+1$-edge partition such that for all edges (u, v) in G,*

$$\sum_{i \in X_u} m_i \neq \sum_{j \in X_v} m_j.$$

In other words, the sum of all elements in the multiset for each vertex (with multiplicity) induces a vertex colouring of G with the elements of M.

An immediate consequence of this is the following lemma:

Lemma 1.4. *Every nice* 3*-colourable graph permits a vertex colouring* 3*-edge partition.*

2. The proof of Theorem 1.1

Because of Lemma 1.4 our approach need only work for non-3-colourable graphs. The key to the proof is the following:

Lemma 2.1. *The vertices of every connected graph which is not* 3*-colourable can be partitioned into* 3 *sets V_0, V_1, V_2 such that*

1. $\forall v \in V_i$, $|N(v) \cap V_{i+1}| \geqslant |N(v) \cap V_i|$, *and*
2. *every vertex in V_i has a neighbour in V_{i+1}.*

In the statement of the lemma and in the rest of the paper, addition on the indices of the V_i's is understood to mean $i \in 0, 1, 2 \bmod 3$. With the lemma in hand it is easy to provide the proof of Theorem 1.1:

Proof of Theorem 1.1. Without loss of generality we assume G is a connected, non-3-colourable graph. We partition $V(G)$ into 3 sets, V_0, V_1, V_2 satisfying the conditions of Lemma 2.1. We label every edge either 0, 1, 2, or 3. We label edges within V_i with i and edges between V_i and V_{i+1} with either i or 3. In this way, vertices in V_i are only incident to edges with labels i, $(i-1) \bmod 3$, and 3. We will ensure that each vertex of V_i is incident to an edge labelled i. Hence, adjacent vertices in distinct V_i will have distinct multisets of labels. (Indeed, in this case, the *sets* of labels are distinct.) By choosing our labelling carefully, we will also ensure that for any two adjacent vertices $u, v \in V_i$, the number of edges with label i incident to u will be distinct from the number of edges with label i incident to v.

Consider the vertices of V_i in some arbitrary order. If v has no internal neighbours, we label all of its edges to V_{i+1} with label i. By Condition 2 of Lemma 2.1, there is at least one such edge. In conjunction with edges internal to V_i, this ensures that every vertex in V_i is incident to an edge with label i.

If v has an internal neighbour, greedily choose an integer, d_v^* between $|N(v) \cap V_i|$ and $2|N(v)\cap V_i|$ which is distinct from d_u^* for all $u \in N(v)\cap V_i$ for which we have already chosen d_u^*. Condition 1 of Lemma 2.1 ensures that there are at least $|N(v) \cap V_i|$ edges from v to V_{i+1}. We assign label i to $d_v^* - |N(v)\cap V_i|$ of them and assign label 3 to the rest. (In this way, we have equated d_v^* to the number of edges incident to v that are labelled i, i.e., the sum of the internal degree and a subset of the edges to V_{i+1}.) Thus, we have ensured that for any two adjacent vertices $u, v \in V_i$, the multiplicity of label i differs in X_u and X_v. This completes the proof.

Proof of Lemma 2.1. A k-cut of G is a partition of the vertices of $V(G)$ into sets $V_1, \ldots, V_k$. A k-cut $V_1, \ldots, V_k$ is *maximum* if the number of edges between partition blocks is maximum over all possible k-cuts. Note that Condition 1 of Lemma 2.1 holds for any maximum 3-cut $K = (V_0, V_1, V_2)$ of G. We will choose a maximum 3-cut such that Condition 2 also holds.

To this end, for every cut $K = (V_0, V_1, V_2)$ define a directed graph $\overrightarrow{G_K} = (V, \overrightarrow{E_K})$, where $\overrightarrow{E_K}$ is defined to consist of

- An arc $\langle v, w\rangle$ for each edge (v, w) with $v \in V_i$, $w \in V_{i+1}$, and
- arcs $\langle v, w\rangle$ and $\langle w, v\rangle$ for any edge (v, w) with $v, w \in V_i$.

Notice that a vertex $v \in V_i$ has outdegree at least 1 if and only if v has a neighbour in $V_i \cup V_{i+1}$. When the cut is maximum, if v has a neighbour in V_i then v has a neighbour in V_{i+1}. Thus, if all vertices in $\overrightarrow{G_K}$ have outdegree at least 1 and K is maximum then Condition 2 of the lemma holds.

Say that u is a descendent of v if there exists a directed path from v to u in $\overrightarrow{G_K}$. For any 3-cut K, let F_K be the set of vertices v for which either

(a) v is on a directed cycle in $\overrightarrow{G_K}$ (which may be a digon), or
(b) v has a descendent u which is on a directed cycle in $\overrightarrow{G_K}$.

Since every vertex in F_K has outdegree at least 1, if $F_K = V(G)$ then Condition 2 holds and the lemma is proved. We choose a maximum 3-cut K maximizing $|F_K|$ over all such cuts. If $F_K \neq V(G)$ we exhibit a maximum 3-cut K' with $|F_{K'}| > |F_K|$, contradicting the maximality of $|F_K|$.

Suppose $F_K \neq V(G)$ and define $K' = (V_0', V_1', V_2')$ by moving every vertex not in F_K to the partition which follows it. In other words:

- If $y \in F_K \cap V_i$ then $y \in V_i'$.
- If $y \in (V(G) - F_K) \cap V_i$ then $y \in V_{i+1}'$.

To establish the maximality of K', it suffices to show that any edge whose endpoints were in distinct blocks of the partition K has endpoints in distinct blocks of K'. Vertices in F_K are in the same block of K' as of K, so we need only consider edges with at least one endpoint in $V(G) - F_K$.

Note that if $y \in V_i - F_K$ then y has no neighbours in V_i by property (a). Note further that if $\langle y, z\rangle \in \overrightarrow{E_K}$ and $z \in F_K$ then $y \in F_K$. Thus we need only consider the following two cases:

- If $\langle y, z\rangle$ is an arc with both endpoints in $V(G) - F_K$ then $y \in V_i$ and $z \in V_{i+1}$, so $y \in V_{i+1}'$ and $z \in V_{i+2}'$. (In this case $\langle y, z\rangle$ is also in $\overrightarrow{G_{K'}}$.)
- If $\langle y, z\rangle \in \overrightarrow{G_K}$ and $y \in F_K$, $z \notin F_K$ then $y \in V_i$ and $z \in V_{i+1}$, so $y \in V_i'$ and $z \in V_{i+2}'$. (In this case, the arc corresponding to (y, z) in $\overrightarrow{G_{K'}}$ is $\langle z, y\rangle$—it is reversed.)

In both cases, the endpoints of (y, z) remain in distinct blocks—thus K' is maximal.

Since G is not 3-colourable $\overrightarrow{G_K}$ contains a digon and hence F_K is not empty. Since G is connected there is an edge (v, w) of G with $v \in F_K$, $w \notin F_K$. By the definition of F_K, $\langle v, w\rangle \in \overrightarrow{E_K}$. To establish that $|F_{K'}| > |F_K|$, we need only show that for all $z \in F_K$, $z \in F_{K'}$, and that $w \in F_{K'}$.

The path (or cycle) showing that a vertex z is in F_K is completely contained in F_K. The only arcs that are reversed in $\overrightarrow{G_{K'}}$ have at least one endpoint not in F_K. Thus the path (or cycle) is unchanged and z is in $F_{K'}$. The arc corresponding to edge (v, w) in $\overrightarrow{E_{K'}}$ is $\langle w, v\rangle$. Since $v \in F_{K'}$, so is w. This completes the proof. $\square$

With minor modifications to these techniques, it can be shown that it is possible to compute a vertex colouring 4-edge partition in polynomial time for any nice graph.

3. The proof of Theorem 1.2

As usual, for a subgraph $H \subseteq G$, let $d_H(v) := |N(v) \cap V(H)|$. We abuse notation by writing $d_W(v) := |N(v) \cap W|$ for any $W \subseteq V(G)$. In proving this theorem, we need to apply the following lemma and corollaries from [1]. The proofs are included here for completeness:

Lemma 3.1. *Let G be a graph and suppose we have chosen, for each vertex v, non-negative integers a_v and b_v such that $a_v < b_v$. Then, precisely one of the following holds:*
(i) *$\exists H \subseteq G$ such that $\forall v$, $a_v \leqslant d_H(v) \leqslant b_v$, or*
(ii) *$\exists A, B \subset V(G)$, $A \cap B = \emptyset$ such that:*

$$\sum_{v \in A} a_v - \sum_{v \in A} d_{G-B}(v) > \sum_{v \in B} b_v.$$

Furthermore, if (ii) *holds, we can find* A *and* B *satisfying* (ii) *such that for all* $w \in A$, *we can choose a subgraph* H *of* G *such that for all* v, $d_H(v) \leqslant b_v$, *which minimizes*

$$\sum_{v \in V} \max(0, a_v - d_H(v))$$

and satisfies

(I) $e \in H$ $\forall e$ *with one endpoint in* A *and no endpoint in* B, *and*
(II) $e \notin H$ $\forall e$ *with one endpoint in* B *and no endpoint in* A.
(III) $d_H(w) < a_w$.

Obviously only one of (i) *or* (ii) *can hold*.

Proof. Choose H such that $d_H(v) \leqslant b_v$ for all v and minimizing $\sum_{v \in V} \max(0, a_v - d_H(v))$. If this sum is zero, then (i) holds. Otherwise, the set $A' = \{v | d_H(v) < a_v\}$ is non-empty. By an H-*alternating path* we mean a path with one endpoint in A' whose first edge is not in H and whose edges alternate between being in H and not being in H.

We let A be those vertices which are the endpoints of some H-alternating path of even length (thus $A' \subseteq A$ since we permit paths of length 0). We let B be those vertices which are the endpoints of some H-alternating path of odd length.

For any H-alternating path P of length > 0, we let H_P be the graph with $E(H_P) = E(H - P) + E(P - H)$. Then, H_P contradicts our choice of H unless either P is odd and one endpoint v of P satisfies $d_H(v) = b_v$, or P is even and both endpoints v of P satisfy $d_H(v) \leqslant a_v$. It follows that:

$$d_H(v) \leqslant a_v \quad \forall v \in A$$

and

$$d_H(v) = b_v \quad \forall v \in B.$$

Thus, A and B are disjoint. Furthermore, (I) and (II) hold by definition. Now, $\sum_{v \in A} d_H(v) < \sum_{v \in A} a_v$ but by (I) and (II),

$$\sum_{v \in A} d_H(v) = \sum_{v \in B} b_v + \sum_{v \in A} d_{G-B}(v).$$

To see that (III) also holds, if $w \in A - A'$, let P be an H-alternating path ending in w and replace H by H_P. This proves the lemma. □

Remark. See the book by Lovasz and Plummer [8] for a proof of a similar theorem which inspired our result.

Corollary 3.2. *Every graph* G *can be partitioned into two subgraphs* G_1 *and* G_2 *such that for all* v, $d_{G_1}(v) \in \left\lfloor \frac{d_G(v)}{2} \right\rfloor, \left\lfloor \frac{d_G(v)}{2} \right\rfloor + 1$.

Proof. Set $a_v := \lfloor d_G(v)/2 \rfloor$, $b_v := \lfloor d_G(v)/2 \rfloor + 1$. For all $A, B \subseteq V$ with $A \cap B = \emptyset$,

$$\begin{aligned}\sum_{v\in A} a_v - \sum_{v\in A} d_{G-B}(v) &= \sum_{v\in A} \left\lfloor \frac{d_G(v)}{2} \right\rfloor - d_{G-B}(v)\\ &\leqslant \sum_{v\in A} \frac{1}{2} d_B(v)\\ &= \sum_{v\in B} \frac{1}{2} d_A(v)\\ &\leqslant \sum_{v\in B} \frac{1}{2} d_G(v) < \sum_{v\in B} b_v. \qquad \square\end{aligned}$$

Corollary 3.3. *Suppose that for some graph F we have chosen, for every vertex v, two integers a_v^- and a_v^+ such that $\frac{d_G(v)}{3} \leqslant a_v^- \leqslant \frac{d_G(v)}{2}$ and $\frac{d_G(v)}{2} \leqslant a_v^+ \leqslant \frac{2d_G(v)}{3} - 1$. Then there is a subgraph H of F such that for every vertex v:*

$$d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}.$$

Proof. For each vertex v, choose either $a_v = a_v^-$, $b_v = a_v^- + 1$ or $a_v = a_v^+$, $b_v = a_v^+ + 1$ and a subgraph H with $d_H(v) \leqslant b_v$ for all v minimizing $\sum_{v\in V} \max(0, a_v - d_H(v))$ over all such choices of a_v and H. We can assume (ii), (I)–(III) of Lemma 3.1 hold as if (i) of that lemma holds, we are done. We will need the following claims:

Claim 3.4. *For all $v \in A$, $a_v - d_{G-B}(v) \leqslant \frac{1}{2} d_B(v)$.*

Claim 3.5. *For all $v \in B$, $b_v \geqslant \frac{1}{2} d_A(v)$.*

Given that these claims hold, we have

$$\sum_{v\in A} a_v \leqslant \sum_{v\in B} b_v + \sum_{v\in A} d_{G-B}(v),$$

a contradiction. So it remains to prove our claims.

Proof of Claim 3.4. We apply (III) to ensure $d_H(v) < a_v$. We can assume $a_v = a_v^+$ as otherwise

$$a_v - d_{G-B}(v) \leqslant \frac{1}{2} d_G(v) - d_{G-B}(v) \leqslant \frac{1}{2} d_B(v),$$

as desired. Also, $d_H(v) > a_v^- + 1$ as otherwise setting $a_v = a_v^-$ contradicts the fact that our choices minimized $\sum_{v\in V} \max(0, a_v - d_H(v))$.

More strongly, $d_{H-B}(v) > a_v^- + 1$ as otherwise setting $a_v = a_v^-$ and deleting $d_H(v) - a_v^- - 1$ edges of H between v and B contradicts our choice of H. Observe that (I) implies that all edges from A to $G - B$ lie in H, so $d_{H-B}(v) = d_{G-B}(v)$.

So,

$$d_{G-B}(v) > \frac{1}{3} d_G(v),$$

which implies

$$d_B(v) < \frac{2}{3} d_G(v)$$

and hence

$$d_{G-B}(v) > \frac{1}{2} d_B(v).$$

So,

$$\begin{aligned} a_v - d_{G-B}(v) &< \frac{2}{3} d_G(v) - d_{G-B}(v) \\ &= \frac{2}{3} d_B(v) - \frac{1}{3} d_{G-B}(v) \\ &\leqslant \frac{2}{3} d_B(v) - \frac{1}{6} d_B(v) = \frac{1}{2} d_B(v). \qquad \square \end{aligned}$$

Proof of Claim 3.5. We can assume that $b_v = a_v^- + 1$ as otherwise $b_v > \frac{1}{2} d_G(v)$ and the claim holds. Furthermore, $d_A(v) > 2b_v$ or the claim holds. Thus, there is a vertex u of A joined to v by an edge not in H. As in the proof of Lemma 3.1, we can augment along an H-alternating path ending in u so as to ensure that $d_H(u) < a_u$ without changing $d_H(v)$, A, B or the fact that $uv \notin E(H)$. Now, we set $a_v = a_v^+$ and choose a set S of $a_v^+ - a_v^- - 1$ vertices of A joined to v by an edge of $G - H$ *including* u. This is possible because $d_B(v) > 2b_v > a_v^+$.

We set $H' = H + \{wv | w \in S\}$ and note that these choices contradict our choice of H. $\square$

With these results in hand, we can prove Theorem 1.2.

Proof of Theorem 1.2. Let G be a graph of minimum degree $\geqslant 1000$. We can greedily colour G so that for each vertex v, the colour $c(v)$ of G is between 0 and $d_G(v)$. We think of $c(v)$ as a pair $(p(v), r(v))$ where

$$p(v) = \left\lfloor \frac{c(v)}{\left\lceil \sqrt{d_G(v)} \right\rceil} \right\rfloor$$

and

$$r(v) = c(v) \bmod \left\lceil \sqrt{d_G(v)} \right\rceil.$$

Applying Corollary 3.2, we can find subgraphs G_1 and G_2 of G such that

$$d_{G_i}(v) \in \left\{ \left\lfloor \frac{d_G(v)}{2} \right\rfloor - 1, \left\lfloor \frac{d_G(v)}{2} \right\rfloor, \left\lfloor \frac{d_G(v)}{2} \right\rfloor + 1 \right\}.$$

We set

$$a_v^-(1) = \left\lfloor \frac{d_G(v)}{4} \right\rfloor - 2p(v) - 1,$$
$$a_v^+(1) = \left\lfloor \frac{d_G(v)}{4} \right\rfloor + 2p(v) + 1.$$

Since $p(v) < \sqrt{d_G(v)}$ and $d_G(v) > 1000$, we see that $a_v^- \geqslant \frac{d_{G_1}(v)}{3}$ and $a_v^+ \leqslant \frac{2d_{G_1}(v)}{3}$.

So, applying Corollary 3.3, we can find a subgraph H_1 of G_1 such that for every vertex v,

$$d_{H_1}(v) \in \left\{a_v^-(1), a_v^-(1) + 1, a_v^+(1), a_v^+(1) + 1\right\}.$$

We set

$$a_v^-(2) = \left\lfloor \frac{d_G(v)}{4} \right\rfloor - 2r(v) - 1$$

and

$$a_v^+(2) = \left\lfloor \frac{d_G(v)}{4} \right\rfloor + 2r(v) + 1$$

and find a subgraph H_2 of G_2 such that for every vertex v,

$$d_{H_2}(v) \in \{a_v^-(2), a_v^-(2) + 1, a_v^+(2), a_v^+(2) + 1\}.$$

We label the edges of H_i with colour i and the remaining edges with label 0. If u and v are not distinguished by our labelling then $d_G(u) = d_G(v)$, $d_{H_1}(u) = d_{H_1}(v)$ and $d_{H_2}(u) = d_{H_2}(v)$ from which it follows that $p(u) = p(v)$ and $r(u) = r(v)$. But, this implies $uv \notin E(G)$ by our choice of the p and r.

References

[1] L. Addario-Berry, K. Dalal, C. McDiarmid, B. Reed, A. Thomason, Vertex colouring edge weights, 2004, submitted for publication.

[2] M. Aigner, E. Triesch, Zs. Tuza, Irregular assignments and vertex-distinguishing edge-colorings of graphs, in: A. Barlotti et al. (Ed.), Combinatorics, vol. 90, Elsevier Science Publishers, New York, 1992, , pp. 1–9.

[3] A.C. Burris, R.H. Schelp, Vertex-distinguishing proper edge colourings, J. Graph Theory 26 (2) (1997) 73–82.

[4] P.N. Balister, O.M. Riordan, R.H. Schelp, Vertex-distinguishing edge colorings of graphs, J. Graph Theory 42 (2) (2003) 95–109.

[5] K. Edwards, The harmonious chromatic number of bounded degree graphs, J. London Math. Soc. 255 (3) (1997) 435–447.

[7] M. Karoński, T. Luczak, A. Thomason, Edge weights and vertex colours, J. Combin. Theory B 91 (2004) 151–157.

[8] L. Lovasz, M.D. Plummer, Matching Theory, Academic Press, New York, 1986.