Oriented trees in digraphs

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July 19, 2012

Abstract

Let $f(k)$ be the smallest integer such that every $f(k)$-chromatic digraph contains every oriented tree of order k. Burr proved $f(k) \leq (k-1)^2$ in general, and conjectured $f(k) = 2k-2$. Burr also proved that every $(8k-9)$-chromatic digraph contains every antidirected tree. We improve both of Burr's bounds. We show that $f(k) \leq k^2/2 - k/2 + 1$ and that every antidirected tree of order k is contained in every $(5k-7)$-chromatic digraph.

We make a conjecture which explains why antidirected trees are easier to handle. It states that if $|E(D)| > (k-2)|V(D)|$, then the digraph D contains every antidirected tree of order k. This is a common strengthening of both Burr's conjecture for antidirected trees and the celebrated Erdős-Sós Conjecture. We note that the analogue of our conjecture for general trees is false, no matter what function $f(k)$ is used in place of $k-2$. We prove our conjecture for antidirected trees of diameter 3, and present some other evidence for it.

Along the way, we show that every acyclic k-chromatic digraph contains every oriented tree of order k and suggest a number of approaches for making further progress on Burr's conjecture.

1 Introduction

We assume the reader is familar with the basic concepts in graph theory; we follow the notation in [3] and suggest readers who need a refresher consult that book.

All the graphs and digraphs we will consider here are *simple*, i.e. they have no loops nor multiple arcs. An *orientation* of a graph G is a digraph obtained from G by replacing every edge uv of G by exactly one of the two arcs uv or vu. An *oriented graph* is an orientation of a graph. Similarly an *oriented tree* (resp. *oriented path*) is an orientation of a tree (resp. path).

A *proper k-colouring* of a digraph is a mapping c from its vertex into $\{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for every arc uv. A digraph is *k-colourable* if it admits a proper k-colouring. The *chromatic number* of a digraph D, denoted $\chi(D)$, is the least integer k such that D is k-colourable. A digraph is *k-chromatic* if its chromatic number equals k.

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The focus of our paper is the following natural question: which digraphs are contained in every n-chromatic digraph? Such digraphs are called *n-universal.*. Clearly an n-universal digraph has at most n vertices, as every tournament (i.e. orientation of a complete graph) on n vertices is n-chromatic. Since there exist n-chromatic graphs with arbitrarily large girth [13], n-universal digraphs must be oriented trees.

The celebrated Gallai-Hasse-Roy-Vitaver Theorem [16, 18, 25, 28] states that every directed path with n vertices is n-universal. However, it is not the case that every oriented tree on n-vertices is n-universal since there exists (regular) tournaments of order $2k-3$ in which every vertex has indegree and outdegree $k-1$. Such a tournament has chromatic number $2k-3$ but has no vertex of out-degree exceeding $k-1$ and thus does not contain the oriented tree S^+_{2k-3} consisting of a vertex dominating $2k-4$ leaves.

Burr [6] considered the function f such that every oriented tree of order k is $f(k)$-universal. He proved $f(k) \leq (k-1)^2$ and conjectured the following bound.

Conjecture 1 (Burr [6])**.** *$f(k) = 2k-2$; i. e., every oriented tree of order k is $(2k-2)$-universal.*

Since a regular tournament of order $2k-1$ does not contain S^+_k, as Burr remarked, the conjectured bound is best possible.

Conjecture 1 is a generalization of Sumner's conjecture that every oriented tree of order k is contained in every tournament of order $2k-2$. Häggkvist and Thomason [17] were the first to show that there was an absolute constant C such that every tournament of order Ck contains every oriented tree of order k. Their bound of Ck was improved to $3k-3$ by El Sahili [12], who used a refinement of an idea in [20]. Recently, Kühn, Osthus, and Mycroft [22] proved that Sumner's conjecture holds for all sufficiently large k. The proof makes extensive use of results and ideas from a recent paper by the same authors [21], in which an approximate version of the conjecture was proved.

Despite considerable interest in Burr's conjecture, until this paper there has been no improvement on the bound on $f(k)$ he obtained over thirty years ago. The only progress has been with respect to a few specific classes of paths. El-Sahili proved [11] that every oriented path of order 4 is 4-universal and that the antidirected path of order 5 is 5-universal. Addario-Berry, Havet, and Thomassé [1] showed that every oriented path of order k whose vertex set can be partioned into two directed paths is k-universal.

In Section 2, we present several approachs that yield bounds on $f(k)$ matching or improving that given by Burr. Our best bound is obtained by proving and then exploiting the fact that Burr's conjecture holds when restricted to acyclic digraph. From this we obtain $f(k) \leq k^2/2 - k/2 + 1$. We hope that these approachs willeventually yield even stronger bounds.

In Section 3, we study the universality of *antidirected trees*; these are oriented trees in which every vertex has in-degree 0 or out-degree 0. Burr [7] showed that every digraph D with at least $4(k-1)|V(D)|$ arcs contains all antidirected trees of order k. From this, it follows that every antidirected tree of order k is $(8k-7)$-universal. We improve this bound to $5k-9$ (for $k \geq 2$) in Subsection 3.1.

We then consider the smallest integer $a(k)$ such that every digraph D with more than $a(k)|V(D)|$ arcs contains every antidirected tree of order k. The above-mentionned result of Burr asserts $a(k) \leq 4k-4$. We conjecture that $a(k) = k-2$.

Conjecture 2. Let D be a digraph. If $|E(D)| > (k-2)|V(D)|$, then D contains every antidirected tree of order k.

The value $k-2$ for $a(k)$ would be best possible. Indeed the oriented tree S^+_k is not contained in any digraph in which every vertex has outdegree $k-2$. It is also tight because the complete symmetric digraph on $k-1$ vertices $\overleftrightarrow{K}_{k-1}$ has $(k-2)(k-1)$ arcs but trivially does not contain any oriented tree of order k.

There is no analogue of Conjecture 2 for non-antidirected trees. Indeed, a bipartite digraph with bipartition (A, B) such that all the arcs have tail in A and head in B contains no directed paths of length

2. Hence for any oriented tree T that is not antidirected and any constant C, there is a digraph D with at least $C \times |V(D)|$ arcs that does not contain T.

Conjecture 2 for oriented graphs implies Burr's conjecture for antidirected trees. Indeed, every $(2k-2)$-critical digraph D is an oriented graph and has minimum degree at least $2k-3$, and hence at least $\frac{2k-3}{2}|V(D)| > (k-2)|V(D)|$ arcs.

Conjecture 2 may be seen as a directed analogue of the well-known Erdős-Sós conjecture.

Conjecture 3 (Erdős and Sós, 1963[14])**.** Let G be a graph. If $|E(G)| > \frac{1}{2}(k-2)|V(G)|$, then G contains every tree of order k .

In fact, Conjecture 2 for symmetric digraphs is equivalent to Conjecture 3. Indeed, consider a graph G and its corresponding symmetric digraph D (the digraph obtained from G by replacing each edge uv by the two arcs uv and vu). Note that G has more than $\frac{1}{2}(k-2)|V(G)|$ edges if and only if $|E(D)| > (k-2)|V(D)|$. Furthermore, if T is a tree and $\vec{T}$ is one of its (two) antidirected orientations, then it is easy to check that G contains T if and only if D contains $\vec{T}$.

Conjecture 3 has been proved in particular cases: when G contains no 4-cycle [26] and for trees with diameter at most 4 [23]. Furthermore, using the Regularity Lemma, Ajtai et al. [2] proved that Conjecture 3 is true for sufficiently large k.

In Subsection 3.2, we settle Conjecture 2 for antidirected trees of diameter at most 3. This allows us to conclude that every antidirected tree of order k and diameter at most 3 is $(2k-3)$-universal. We then strengthen this result showing that such trees are actually $(2k-4)$-universal.

2 Approaches for upper bounds on $f(k)$

2.1 Constructing trees iteratively

Let T be an oriented tree. The *in-leaves* (resp. *out-leaves*) of T are the vertices v of T such that $d^+_T(v) = 1$ and $d^-_T(v) = 0$, (resp. $d^+_T(v) = 0$ and $d^-_T(v) = 1$). The set of out-leaves (resp. in-leaves) of T is denoted by $\mathrm{Out}(T)$ (resp. $\mathrm{In}(T)$) and its cardinality is denoted by $\mathrm{out}(T)$ (resp. $\mathrm{in}(T)$).

An *out-star* is an oriented tree T such that $T - \mathrm{Out}(T)$ is a single vertex x. The out-star of order k is denoted S^+_k. An *in-star* is defined analagously. The in-star of order k is denoted S^-_k. In this paper, *star* is used to mean an out-star or an in-star.

Clearly any digraph D with $(k-1)|V(D)|$ arcs contains both an in-star and out-star of order k. Thus, stars of order k are $2k-1$-universal. In this section, we show that every tree T is $g(T)$ universal for some function g which depends only on the order of T and how "close" it is to a star.

To this end, we let $\mathrm{st}(T)$ be the minimum number of successive removals of either the set of in-leaves or the set of out-leaves which reduces the oriented tree to a single vertex. Since each such removal removes one or two edges of a longest path, we have $\lceil \mathrm{diam}(T)/2 \rceil \leq \mathrm{st}(T) \leq \mathrm{diam}(T)$.

We will show in this section that every oriented tree T of order k is $[(2k-3-\mathrm{st}(T))\,\mathrm{st}(T)+2]$-universal We begin with the following simple lemma.

Lemma 4. *Let D be a digraph with both minimum in-degree and minimum out-degree at least $k-1$, and let T be a tree of order k. For any vertex x of D and vertex t of T, D contains a copy of T in which x corresponds to t.*

Proof. We prove the result by induction on k, the result holding trivially when $k = 1$. Assume now that $k \geq 2$. Let v be a leaf of T distinct from t. By symmetry, we may assume that v is an out-leaf. Let u be its in-neighbour in T. By the induction hypothesis, D contains a copy T' of $T - v$ in which x corresponds to t. Let y be the vertex corresponding to u in T'. Since $d^+(y) \geq k-1$, there is an out-neighbour z of y not in $V(T')$. Adding the vertez z and the arc yz to T' yields the desired copy of T. □

We will need the following slight strengthening of this lemma.

Lemma 5. *Let D be an oriented graph with both minimum in-degree and minimum out-degree at least $k-2$ and T a tree of order k. If T has two out-leaves which do not have a common neighbour, then D contains T.*

Proof. Let v_1 and v_2 be two out-leaves of T with v_1 adjacent to u_1 and v_2 adjacent to $u_2 \neq u_1$. By Lemma 4, D contains a copy T' of $T - v_1$. Let x_1, x_2, and y be the vertices of T' corresponding to u_1, u_2 and v_2 in $T - v_1$, respectively. If x_1 has an out-neighbour z_1 in $V(D) \setminus V(T')$, then adding z_1 and the arc x_1z_1 to T' yields a copy of T.

So we may assume that all the out-neighbours of x_1 are in $V(T')$. Since $d^+(x_1) \geq k-2$, x_1 dominates all the vertices of $T' - x_1$. In particular, it dominates x_2 and y. Hence the tree T'' obtained from T' by removing the arc x_2y and adding the arc x_1y is a copy of $T - v_2$. Now x_2 has out-degree at least $k+2$, and it does not dominate x_1 because D is an oriented graph. So x_2 has an out-neighbour z_2 in $V(D) \setminus V(T'')$. Thus adding z_2 and the arc x_2z_2 to T'' yields a copy of T. □

We now show how to apply this result to obtain the main result of this section. We begin with 4 definitions and a simple auxiliary result.

A (di)graph is *k-degenerate* if all its sub(di)graphs have a vertex of degree at most k. It is well-known and easy to show that every k-degenerate (di)graph is $(k+1)$-colourable.

A (di)graph is *k-critical* if its chromatic number is k and all its proper sub(di)graphs are $(k-1)$-colourable. It is folklore that every k-critical graph has minimum degree at least $k-1$.

The *average degree* of a (di)graph G, denoted $\mathrm{Ad}(G)$, is $\dfrac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \dfrac{2|E(G)|}{|V(G)|}$. The *maximum average degree* of G, denoted $\mathrm{Mad}(G)$, is $\max\{\mathrm{Ad}(H) : H \subseteq G\}$.

Lemma 6. *If G is a graph of maximum average degree at most k, where k is an integer with $k \geq 3$, then $\chi(G) \leq k$ or G contains a complete graph on $k+1$ vertices.*

Proof. If $\chi(G) > k$, then G contains a $(k+1)$-critical graph H. Since $\delta(H) \geq k$ and $\mathrm{Mad}(G) \leq k$, the subgraph H must be k-regular. Because $\chi(H) = k+1$, by Brooks' Theorem, H is a complete graph on $k+1$ vertices. □

We are now ready to exploit Lemma 5. We prove:

Lemma 7. *For $k \geq 3$, let T be an oriented tree of order k other than S_k^+. If $T - \mathrm{Out}(T)$ is l-universal, then T is $(l+2k-4)$-universal.*

Proof. Since $T \neq S_k^+$, it follows that $T - \mathrm{Out}(T)$ has more than one vertex, and thus $l \geq 2$. If $\mathrm{Out}(T) = \emptyset$, then the result holds trivially, so we assume $\mathrm{out}(T) \geq 1$.

Let D be an $(l+2k-4)$-chromatic digraph. Without loss of generality, we may assume that D has no subgraph which is $(l+2k-4)$-chromatic and hence is connected. Let S be the set of vertices of D with out-degree at most $k-2$.

Assume first that $\chi(D-S) \geq l$, so $D-S$ contains a copy T' of $T - \mathrm{Out}(T)$. Let $v_1, v_2, \ldots, v_p$ be the out-leaves of T, and let $w_1, w_2, \ldots, w_p$ be their respective in-neighbours in T. Now for $1 \leq i \leq p$, since the out-degree of w_i', the vertex corresponding to w_i in T', is at least $k-1$ in D, one can find an out-neighbour v_i' of w_i' in $V(D) \setminus (V(T') \cup \{v_j' \mid 1 \leq j < i\})$. Hence D contains T.

Assume now that $\chi(D-S) < l$. Since $\chi(D) = l+2k-4$, we must have $\chi(D[S]) \geq 2k-3$. If H is a subdigraph of $D[S]$, then $\sum_{v \in V(H)} d(v) = 2E(H) \leq 2\sum_{v \in V(H)} d_D^+(v) \leq (2k-4) \times |V(H)|$. Hence $\mathrm{Mad}(D[S]) \leq 2k-4$. Thus by Lemma 6, $D[S]$ contains a tournament R of order $2k-3$. Furthermore, since the out-degree of a vertex in R is at most its out-degree in $D[S]$ and thus is bounded by $k-2$, every vertex of R has both in- and out-degree equal to $k-2$ in R. Since all vertices in R have out-degree at most $k-2$ in D, each vertex of R has no out-neighbour in $V(D) \setminus V(R)$. Now, since D is connected, there is an arc xy with $x \in V(D) \setminus V(R)$ and $y \in V(R)$.

If T contains an in-leaf v, then let u be its out-neighbour in T. By Lemma 4, R contains a copy of $T - v$ such that u corresponds to y. This copy together with the vertex x and the arc xy is a copy of T in D.

If T contains no in-leaf, then it contains only out-leaves. Moreover, since $T \neq S_k^+$, then T has two leaves which have distinct neighbours. Thus by Lemma 5, R contains T. □

The main result of the section now follows easily.

Proposition 8. *Every oriented tree T of order k is $[(2k-3-\mathrm{st}(T))\,\mathrm{st}(T)+2]$-universal.*

Proof. Let $T_0, T_1, \dots, T_{\mathrm{st}(T)}$ be a sequence of oriented trees such that $T_0 = T$, $T_{\mathrm{st}(T)}$ is a tree with a unique vertex and, for $1 \leq i \leq \mathrm{st}(T)$, $T_i = T_{i-1} - \mathrm{Out}(T_{i-1})$ or $T_i = T_{i-1} - \mathrm{In}(T_{i-1})$. Then $T_{\mathrm{st}(T)-1}$ is a star and thus is $(2|T_{\mathrm{st}(T)-1}|-2)$-universal. By successive application of Lemma 7, T is contained in every digraph of chromatic number at least Σ with $\Sigma = 2|T_0| - 4 + 2|T_1| - 4 + \cdots + 2|T_{\mathrm{st}(T)-2}| - 4 + 2|T_{\mathrm{st}(T)-1}| - 2 = 2\sum_{i=0}^{\mathrm{st}(T)-1} |T_i| - 4\,\mathrm{st}(T) + 2$. Now for all $0 \leq i \leq \mathrm{st}(T)-1$, $|T_i| \leq k - i$, so $\Sigma \leq (2k-3-\mathrm{st}(T))\,\mathrm{st}(T)+2$. Hence T is $[(2k-3-\mathrm{st}(T))\,\mathrm{st}(T)+2]$-universal. □

Proposition 8 implies directly a slightly better bound on $f(k)$ than that obtained by Burr [6].

Corollary 9. *Every oriented tree T of order k is (k^2-3k+4)-universal.*

Proof. Let T be a tree of order k. If T is a directed path, then it is k-universal by the Gallai-Hasse-Roy-Vitaver Theorem. If T is not a directed path, then $\mathrm{st}(T) \leq k-2$. So, by Proposition 8, T is (k^2-3k+4)-universal. □

Our results show that if we want to determine if an oriented tree T must exist in digraphs of sufficiently large chromatic number, it would be useful to determine $\mathrm{st}(T)$. However, it is not clear if this can be done efficiently.

Problem 10. What is the complexity of determining $\mathrm{st}(T)$ for a given an oriented tree T?

2.2 Decomposing digraphs into bikernel-perfect subdigraphs

Let D be a digraph. We say that a set S of vertices of D is *dominating* if every vertex v in $V(D) \setminus S$ is dominated by a vertex in S. We say that S is *antidominating* if every vertex v in $V(D) \setminus S$ dominates a vertex in S. A dominating stable set is called a *kernel* and an antidominating stable set an *antikernel*. If every induced subdigraph of D has a kernel (resp. antikernel), then D is said to be *kernel-perfect* (resp. *antikernel-perfect*). A digraph which is both kernel- and antikernel-perfect is said to be *bikernel-perfect*.

Theorem 11. *Every oriented tree of order k is contained in every k-chromatic bikernel-perfect digraph.*

Proof. We assume the theorem is false and consider a counterexample T of minimum order. We let k be this order and note that trivially k is at least 2.

We let D be a k-chromatic bikernel-perfect digraph, let v be a leaf of T, and let w the unique neighbour of v in T. By symmetry, we may assume that $v \to w$. Since D is bikernel-perfect, it has a kernel S. The digraph $D - S$ has chromatic number at least $(k-1)$, so by induction it contains a copy T' of $T - v$. Now by the definition of a kernel, the vertex w' in T' corresponding to w is dominated by a vertex v' of S. Hence D contains T. But since this holds for all D, T is not a counterexample to the theorem, which is a contradiction. □

Several classes of bikernel-perfect digraphs are known. It is easy to show that symmetric digraphs are bikernel-perfect. Richardson [24] proved that acyclic digraphs and more generally, digraphs without directed cycles of odd length are also bikernel-perfect. Several extensions of Richardson's Theorem have been obtained [8, 9, 10, 15]. Sands, Sauer and Woodrow [27] showed that a digraph whose arcs may be partitionned into two posets is bikernel-perfect.

We could attempt to find better upper bounds on $f(k)$ by proving that every digraph with sufficiently high chromatic number contains an acyclic (or more generally bikernel-perfect) k-chromatic subdigraph.

Problem 12. What is the minimum integer $g(k)$ such that every $g(k)$-chromatic digraph has an acyclic k-chromatic subdigraph?

What is the minimum integer $g'(k)$ such that every $g'(k)$-chromatic digraph has a bikernel-perfect k-chromatic subdigraph?

Of course, Theorem 11 implies $f(k) \leq g'(k)$ while trivially $g'(k) \leq g(k)$.

Proposition 13. $g(k) \leq k^2 - 2k + 2$.

Proof. Let D be a (k^2-2k+2)-chromatic digraph. Let $v_1, v_2, \ldots, v_l$ be an ordering of the vertices of D. Let D_1 and D_2 be the digraphs with vertex set $V(D)$ and edge-sets $E(D_1) = \{v_iv_j \in E(D), i < j\}$ and $E(D_2) = \{v_iv_j \in E(D), i > j\}$. Clearly, D_1 and D_2 are acyclic and $\chi(D_1) \times \chi(D_2) \geq \chi(D) = k^2 - 2k + 2$. Hence either D_1 or D_2 has chromatic number at least $\left\lceil \sqrt{k^2 - 2k + 2} \right\rceil = k$. □

The above proposition implies directly $f(k) \leq k^2 - 2k + 2$. We now give a better upper bound for $f(k)$.

Theorem 14. $f(k) \leq k^2/2 - k/2 + 1$.

Proof. Let us prove that $f(k) \leq f(k-1) + k - 1$. Then an easy induction will give the result as $f(1) = 1$.

Let D be an $(f(k-1) + k - 1)$-chromatic digraph and T be an oriented tree of order k. Let A be a maximal acyclic induced subdigraph of D. If $\chi(A) \geq k$, then by Theorem 11, A contains T, so D contains T. If $\chi(A) \leq k-1$, then $\chi(D - A) \geq f(k-1)$. Let v be a leaf of T. The digraph $D - A$ contains $T - v$. Now, by maximality of A, for every vertex x of $D - A$, there are vertices y and z of A such that xy and zx are arcs. So we can extend $T - v$ to T by adding a vertex to A. □

Another approach to bounding $f(k)$ would be to prove the existence of a dominating set with not too large chromatic number in any n-chromatic digraph.

Problem 15. What is the minimum integer $h(n)$ such that every n-chromatic digraph has an $h(n)$-chromatic dominating set?

2.3 Exploiting acyclic subdigraphs

Let D be a digraph. An *acyclic partition* of D is a partition of its vertex set $(V_1, V_2, \ldots, V_p)$ such that the digraph $D[V_i]$ induced by each of the V_i is acyclic. The *acyclic number* of D, denoted $ac(D)$, is the minimum number of parts of an acyclic partition of D. Note that a proper colouring is an acyclic partition because a stable set is acyclic. So $\chi(G) \geq ac(G)$.

Theorem 16. *If T is an oriented tree with vertices $v_1, v_2, \ldots, v_k$ and D is a digraph with acyclic number k, then for any acyclic partition of D into k parts $V_1, V_2, \ldots, V_k$, D contains a copy of T such that $v_i \in V_i$ for every $1 \leq i \leq k$.*

Proof. We prove the result by induction on k, the result being trivial for $k = 1$. Let v be a leaf of T. Since we are free to permute the indices of the vertices of T, provided we permute the indices of the parts of the partition using the same permutation, we may assume that $v = v_k$ and the neighbour of v_k in T is v_{k-1}. Moreover, by symmetry, we may assume that $v_{k-1} \to v_k$. Let us now consider $D' = D[V_1 \cup \cdots \cup V_{k-1}]$. Obviously $ac(D') = k - 1$, so $(V_1, V_2, \ldots, V_{k-1})$ is an acyclic partition of D' in $ac(D')$ sets. Hence, by the induction hypothesis, D' contains at least one copy of $T' = T - v_k$ such that $v_i \in V_i$ for all $1 \leq i \leq k - 1$.

Let S be the set of all those vertices of V_{k-1} that correspond to v_{k-1} in some such copy of T' in D'. We will show that there is a vertex s of S which dominates some vertex in V_k, which gives the result. Suppose for a contradiction that no vertex of S dominates a vertex of V_k. Then $D[V_k \cup S]$ is acyclic. Let us consider $D'' = D' \setminus S$. Then $ac(D'') = k - 1$. Indeed an acyclic partition of D'' in less than $k - 1$ sets together with $S \cup V_k$ would be an acyclic partition of D in less than k sets which is impossible. In particular, $S \neq V_{k-1}$. So $(V_1, \ldots, V_{k-2}, V_{k-1} \setminus S)$ is an acyclic partition of D'' in $ac(D'')$ sets. Thus, by the induction hypothesis, D'' contains a copy of T' such that $v_i \in V_i$ for all $1 \leq i \leq k - 2$ and $v_{k-1} \in V_{k-1} \setminus S$. But this contradicts the definition of S. $\square$

Theorem 16 and Theorem 11 yield that $f(k) \leq (k-1)^2 + 1$. Indeed let D be a $((k-1)^2+1)$-chromatic digraph D and T be an oriented tree of order k. If $ac(D) \geq k$, by Theorem 16, D contains T. If not, in an acyclic partition into $ac(D)$ sets, one of the sets induces an acyclic digraph with chromatic number at least k and by Theorem 11, D contains T.

3 The universality of antidirected trees

In [7], Burr proved that every antidirected tree of order k is contained in every digraph D with at least $4(k-1)|V(D)|$ arcs. This implies trivially that every antidirected tree of order k is $(8k-7)$-universal because every $(8k-7)$-critical digraph D has minimum degree at least $8k-8$ and thus has at least $4(k-1)|V(D)|$ arcs.

In this section, we will first strengthen Burr's result by showing that every antidirected tree of order k is $(5k-9)$-universal. We then settle Conjecture 2 for antidirected trees of diameter at most 3 which implies that such trees are $2k-3$ universal. We then strengthen the latter result, by proving that every such tree is $(2k-4)$-universal.

3.1 An improved upper bound

Let T be an antidirected tree. Let $V^+(T)$ (resp. $V^-(T)$) be the set of vertices with in-degree (resp. out-degree) 0 in T. Clearly $(V^-(T), V^+(T))$ is a partition of $V(T)$. We set $m(T) = \max\{|V^+(T)|, |V^-(T)|\}$.

Theorem 17. *If T is an antidirected tree and $D = (V, E)$ is a digraph with at least $(4m(T)-4)|V|$ arcs, then D contains T.*

This theorem is a slight strengthening of Burr's result [7]. Its proof is based on the following three lemmas, the first two of which were proved and used by Burr.

Lemma 18 (Burr [7])**.** *Let $G = (V, E)$ be a bipartite graph and p be an integer. If $|E| \geq p|V|$, then G has a subgraph with minimum degree at least $p + 1$.*

Remark 19. This result is tight : for any $\epsilon = \frac{p}{m+p} > 0$, the complete bipartite graph $K_{p,m}$ has $pm = p|V|(1-\epsilon)$ edges but every subgraph has minimum degree at most p.

Let (A, B) be a bipartition of the vertex set of a digraph D. We denote by $E(A, B)$ the set of arcs with tail in A and head in B and by $e(A, B)$ its cardinality.

Lemma 20 (Burr [7])**.** Every digraph D contains a partition (A, B) such that $e(A, B) \geq |E(D)|/4$.

Lemma 21. *Let T be an antidirected tree and $D = ((A, B), E)$ be a bipartite graph such that every vertex in A has out-degree at least $m(T)$ and every vertex in B has in-degree at least $m(T)$. Then D contains T.*

Proof. Let us show by induction on $|T|$ that one may find a copy of T such that every vertex of $V^+(T)$ (resp. $V^-(T)$) is in A (resp. B).

Let v be a leaf of T. By symmetry, we may assume that v is an in-leaf so $v \in V^+(T)$. Let u be the out-neighbour of v in T. Then u is in $V^-(T)$. Furthermore, $T - v$ satisfies $m(T - v) \leq m(T)$, so one can find a copy T' of $T - v$ such that every vertex of $V^+(T - v)$ (resp. $V^-(T - v)$) is in A (resp. B). In particular, u is in B. Now u has at least $m(T)$ in-neighbours in A. Moreover, since $V^+(T - v) < m(T)$ one of these in-neighbors is not in T'. Adding this in-neighbour to T' yields the desired copy of T. □

Proof of Theorem 17. By Lemma 20, D contains a spanning bipartite subdigraph $D' = ((A, B), E(A, B))$ with at least $(m(T) - 1)|V|$ edges. By Lemma 18, D' has a bipartite subdigraph $D'' = ((A'', B''), E'')$ such that every vertex of A'' has out-degree at least $m(T)$ and every vertex of B'' has in-degree at least $m(T)$. Hence, by Lemma 21, D'' (and so D) contains T. □

Corollary 22. *Every antidirected tree T is $(8m(T) - 7)$-universal.*

Proof. Every $(8m(T) - 7)$-chromatic digraph D contains an $(8m(T) - 7)$-critical digraph D' which has minimum degree at least $8m(T) - 8$. So D' has at least $(4m(T) - 4)|V(D')|$ arcs. Hence, by Theorem 17, D' (and so D) contains T. □

Note that Corollary 22 is rather good when $m(T)$ is close to $|T|/2$. We will now prove an auxiliary result which is helpful when $m(T)$ is much larger than $|T|/2$.

Lemma 23. *If T is an antidirected tree, then T has at least $Exc(T) = |V^+(T)| - |V^-(T)|$ in-leaves.*

Proof. We proceed by induction on the order of T.

Note that if $Exc(T) \leq 0$, the result is trivial. Suppose now that $Exc(T) > 0$. Let v be a leaf of T.

If v is an in-leaf, then $Exc(T - v) = Exc(T) - 1$. By induction $T - v$ has $Exc(T) - 1$ in-leaves. These leaves and v are the $Exc(T)$ in-leaves of T.

If v is an out-leaf then $Exc(T - v) = Exc(T) + 1$. By induction $T - v$ has $Exc(T) + 1$ in-leaves and at most one of them dominates v. So T has at least $Exc(T)$ in-leaves. □

Theorem 24. *If T is an antidirected tree of order k other than a star, then T is $(10k - 8m(T) - 11)$-universal.*

Proof. By symmetry, we may assume that $Exc(T) \geq 0$. Let F be a set of $Exc(T)$ in-leaves and U be the antidirected tree $T - F$. Then $Exc(U) = 0$, so $m(U) = |U|/2 = k - m(T)$. Hence, by Corollary 22, U is $(8k - 8m(T) - 7)$-universal. Now, by Lemma 7, T is $(10k - 8m(T) - 11)$-universal. □

Corollary 25. *Every antidirected tree T of order $k \geq 3$ is $(5k - 9)$-universal.*

Proof. If T is a star, then it is $(2k - 2)$-universal, which is more than we need.

If T is not a star, then Corollary 22 and Theorem 24 yield that T is $(\min\{8m(T) - 7; 10k - 8m(T) - 11\})$-universal. The first function increases with $m(T)$ and the second decreases with $m(T)$. They are equal when $m(T) = \frac{5}{8}k - \frac{1}{4}$. In this case, the value of the two functions is $5k - 9$. □

3.2 Antidirected trees of diameter 3

In this subsection, we provide evidence for Conjecture 2, by proving it holds for antidirected trees of diameter at most 3.

We note that it is easy to show that Conjecture 2 holds for antidirected trees of diameter 2 because there are only two antidirected trees of order k and diameter 2: the stars S_k^+ and S_k^-.

Proposition 26. *Let D be a digraph. If $|E(D)| > (k-2)|V(D)|$, then D contains S_k^+ and S_k^-.*

Proof. Let D be a digraph with more than $(k-2)|V(D)|$ arcs. Since $\sum_{v\in V(D)} d^+(v) = E(D) > (k-2)|V(D)|$, D contains a vertex of out-degree at least $k-1$. So it contains S_k^+. Similarly, D contains S_k^-. □

Henceforth, we restrict our attention to antidirected trees of diameter 3. An antidirected tree of order k and diameter 3 is made of a central arc uv such that u dominates the $\mathrm{out}(T)$ out-leaves of T and v is dominated by the $\mathrm{in}(T)$ in-leaves of T. Note that $\mathrm{out}(T) \geq 1$ and $\mathrm{in}(T) \geq 1$, so $k = \mathrm{out}(T)+\mathrm{in}(T)+2 \geq 4$.

Lemma 27. *Let D be a digraph, T an antidirected tree of diameter 3 and $uv \in E(D)$. If*

a) $d^+(u) \geq k-1$ and $d^-(v) \geq \mathrm{in}(T)+1$, or

b) $d^+(u) \geq k-2$, $d^-(v) \geq \mathrm{in}(T)+1$ and $N^-(v) \not\subset N^+(u) \cup \{u\}$,

then D contains T.

Proof. Set $\mathrm{in}(T) = p$. Since its in-degree is at least $p+1$, the vertex v has at least p in-neighbours $v_1, v_2, \dots, v_p$ distinct from u with $v_1 \in N^-(v) \setminus N^+(u) \cup \{u\}$ in case b). Since $d^+(u) \geq k-1$ or $v_1 \notin N^+(u)$, the vertex u has $k-2-p = \mathrm{out}(T)$ out-neighbours in $V(D) \setminus \{v, v_1, \dots, v_p\}$. Hence D contains T. □

We will now prove a slight strengthening of Conjecture 2 for antidirected trees of diameter 3. In what follows we write $\vec{K}_k$ for the "complete symmetric" directed graph of order k, with k vertices and a directed edge in each direction between each pair of vertices.

Theorem 28. *Let D be a connected digraph. If $|E(D)| \geq (k-2)|V(D)|$ and $D \neq \vec{K}_{k-1}$, then D contains every antidirected tree of order k and diameter 3.*

Proof. We prove the result by induction on $|V(D)|$. Fix an antidirected tree T of order k and diameter 3. Let V^+ (resp. V^-) be the set of vertices of D of out-degree (resp. in-degree) at least $k-1$.

Suppose first that $V^+ = V^- = \emptyset$. Then every vertex v satisfies $d^+(v) = d^-(v) = k-2$. If D is not $\vec{K}_{k-1}$, then it is not complete symmetric and has at least k vertices. Thus there exist three vertices u, v and v_1 such that $uv \in E(D)$, $v_1v \in E(D)$ and $uv_1 \notin E(D)$. In this case u and v satisfy the condition b) of Lemma 27, and hence D contains T.

It remains to consider the case that at least one of V^+, V^- is non-empty. By symmetry, we may assume that $V^+ \neq \emptyset$. If $V^- = \emptyset$, then every vertex has in-degree $k-2$. Picking a vertex $u \in V^+$ and one of its out-neighbours v, since $k-2 \geq \mathrm{in}(T)$, Lemma 27 gives the result. We may therefore assume that V^+ and V^- are both nonempty.

Let u be a vertex of out-degree at least $k-1$ and v an out-neighbour of u. If $d^-(v) \geq \mathrm{in}(T)+1$, then Lemma 27 gives the result. So we may assume that every out-neighbour of u has in-degree at most $\mathrm{in}(T)$. In particular, the set V_1 of vertices of D with in-degree at most $\mathrm{in}(T)$ has cardinality at least $k-1$. Analogously, we may assume that the set V_2 of vertices of D with out-degree at most $\mathrm{out}(T)$ has cardinality at least $k-1$.

Suppose that $V_1 \cap V_2$ contains some vertex v. Then $d(v) \leq \mathrm{in}(T)+\mathrm{out}(T) = k-2$. Hence $|E(D-v)| \geq (k-2)|V(D-v)|$ and, by the induction hypothesis, T is contained in $D-v$ (and so in D) unless

$D - v = \vec{K}_{k-1}$. But in this case, $d(v) = k-2$ and it is easy to see that D contains T. Hence we may assume that $V_1 \cap V_2 = \emptyset$.

Now suppose that there are $v_1 \in V_1$ and $v_2 \in V_2$ such that $v_1 v_2$ is not an arc of D. Then consider the digraph D' obtained by replacing the two vertices v_1 and v_2 by a vertex t dominating the out-neighbours of v_1 and dominated by the in-neighbours of v_2. The digraph D' has one vertex less than D and at most $d^-(v_1) + d^+(v_2)$ arcs less than D (the $d^-(v_1)$ entering v_1, the $d^+(v_2)$ leaving v_2 and $v_1 v_2 \notin E(D)$). Now $d^-(v_1) + d^+(v_2) \leq \mathrm{in}(T) + \mathrm{out}(T) = k-2$, so $|E(D')| \geq (k-2)|V(D')|$. If $D' \neq \vec{K}_{k-1}$, by the induction hypothesis, D' contains a copy of T. This copy may be transformed into a copy of T in D, by replacing t by v_1 (resp. v_2) if t is a source (resp. a sink) in T. If $D' = \vec{K}_{k-1}$ then $D - \{v_1, v_2\} = \vec{K}_{k-2}$. Since $d^+(v_1) \geq \mathrm{out}(T) + 1$ and $d^-(v_2) \geq \mathrm{in}(T) + 1$, one can again easily see that D contains T.

The only case which remains is that for all $u \in V_1$ and $v \in V_2$, uv is an arc of C. In this case, every vertex $u \in V_1$ has out-degree at least $k-1$ and every vertex $v \in V_2$ has in-degree at least $k-1$. Applying Lemma 27 to any edge uv with $u \in V_1$ and $v \in V_2$ then shows that D contains T. □

Theorem 28 implies that every connected digraph D with minimum degree at least $2k-4$ other than $\vec{K}_{k-1}$ contains every antidirected tree of order k and diameter 3. In particular, this is the case if D is $(2k-3)$-critical. Hence antidirected trees of order k and diameter 3 are $(2k-3)$-universal. We will now improve slightly this result by showing that such trees are $(2k-4)$-universal.

Proposition 29. *Every oriented graph with minimum degree at least $2k-5$ contains every antidirected tree of order k of diameter 3.*

Proof. Suppose for a contradiction that there is some antidirected tree T of order k and diameter 3 which is not contained in some oriented digraph D with minimum degree $2k-5$.

<u>Assume first that $k = 4$.</u> Then T is the antidirected path of length 3.

If D contains no vertex of out-degree 2 then consider any vertex x of out-degree at least 3. By Lemma 27, each out-neighbour of x has in-degree 1 and thus out-degree at least 2 and so at least 3. Then the oriented graph induced by the vertices of out-degree at least 3 contains a vertex of in-degree at least 3. So there is an arc uv such that $d^+(u) \geq 3$ and $d^-(v) \geq 3$, and it follows from Lemma 27 that D contains T, a contradiction.

If D contains some vertex of out-degree exactly 2 then let a be some such vertex and let b and c be its out-neighbours. Vertex b (resp. c) has no in-neighbour u not in $\{a, c\}$ (resp. $\{b, c\}$) for otherwise (u, b, a, c) (resp. (u, c, a, b)) is a copy of T. We claim that there is no arc between b and c. Indeed, suppose there is such an arc, say bc. Since $d(b) \geq 3$, b has a neighbour u distinct from a and c. By the above observation, u is an out-neighbour of b, so (a, c, b, u) is a copy of T, a contradiction.

It follows that $d^-(b) = d^-(c) = 1$ so $d^+(b) = d^+(c) = 2$.

Hence the oriented graph induced by the vertices of out-degree 2 contains a vertex of in-degree at least 2. So there is an arc uv such that $d^+(u) \geq 2$ and $d^-(v) \geq 2$. Moreover by the above claim, $N^-(v) \cap N^+(u) = \emptyset$. Together with Lemma 27, this again yields a contradiction.

<u>Assume now that $k \geq 5$.</u>

By symmetry, we may assume that $\mathrm{in}(T) \leq \mathrm{out}(T)$, so $\mathrm{in}(T) \leq k-4$. Let V^+ be the set of vertices of out-degree at least $k-1$. We claim that $V^+ = \emptyset$; suppose for a contradiction that $V^+ \neq \emptyset$.

If there is no arc uv with $u \in V^+$ and $v \notin V^+$, then each vertex of V^+ has all its out-neighbours in V^+. In this case, the digraph D^+ induced by V^+ has at least $|V^+|(k-1)$ arcs and thus has at least a vertex v of in-degree $k-1$ in D^+. Let u be any in-neighbour of v in D^+. As $d^+(u) \geq k-1$, applying Lemma 27 to uv yields a contradiction. Hence we may assume that there is an arc uv with $u \in V^+$ and $v \notin V^+$, in which case $d^+(v) \leq k-2$ and so $d^-(v) \geq k-3$. Since $\mathrm{in}(T) \leq k-4$, applying Lemma 27 to uv again gives a contradiction. This proves the claim that $V^+ = \emptyset$.

Since $V^+ = \emptyset$, every vertex has out-degree at most $k-2$ and thus in-degree at least $k-3$. We claim that there is a vertex u of out-degree $k-2$ dominating a vertex of in-degree at least $k-2$. If this is not the case then every vertex of out-degree $k-2$ has all its out-neighbours of in-degree at most $k-3$ and thus out-degree at least $k-2$. This implies that V_2, the set of vertices of out-degree $k-2$, has no outgoing arcs and thus there is a vertex v in V_2 with in-degree at least $k-2$ in $D[V_2]$. Picking any in-neighbour u of v in V_2, we obtain the desired vertices, a contradiction.

Finally, by Lemma 27 and our assumption that D does not contain T, for every out-neighbour w of u, we must have that $N^-(w) \subset N^+(u) \cup \{u\}$. Since each vertex has in-degree at least $k-3$ and v has in-degree $k-2$, the digraph $D[N^+(u)]$ has at least $(k-2)(k-4)+1 > \binom{k-2}{2}$ arcs which is impossible since D is an oriented graph.

□

Note that the analogue of Proposition 29 does not holds for digraphs (rather than oriented graphs). Indeed, there are connected digraphs such that $d(v) \geq 2k-5$ for every vertex v that do not contain every antidirected tree of order k of diameter 3. For example, let $G = ((A,B),E)$ be a regular bipartite graph of degree $k-3$. Let D the digraph obtained from G by orienting all the edges from A to B and adding, for each $a \in A$ (resp. $b \in B$) a copy of $\vec{K}_{k-2}$ dominating a (resp. dominated by b). One can easily check that for every vertex v, $d^+(v) + d^-(v) \geq 2k-5$ and that D does not contain the antidirected tree of order k and diameter 3 with one in-leaf.

We close by noting the following consequence of Proposition 29.

Corollary 30. *Every antidirected tree of order k and diameter 3 is $(2k-4)$-universal.*

Proof. Let D be a $(2k-4)$-chromatic digraph. Then D contains a $(2k-4)$-critical oriented graph D', in which every vertex has degree at least $2k-5$. Hence D', and so D, contains every antidirected tree of order k of diameter 3, by Proposition 29. □

Corollary 30 and Proposition 29 are tight. Indeed a regular tournament of order $2k-5$ is $(2k-5)$-chromatic and is an oriented graph with minimum degree $2k-6$ but does not contain the antidirected tree with $k-3$ out-leaves because no vertex has out-degree $k-2$ or more.

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