

CONVERGENCE OF ODD-ANGULATIONS VIA SYMMETRIZATION OF LABELED TREES

LOUIGI ADDARIO-BERRY AND MARIE ALBENQUE

ABSTRACT. Fix an odd integer $p \geq 5$. Let M_n be a uniform p -angulation with n vertices, endowed with the uniform probability measure on its vertices. We prove that there exists $C_p \in \mathbb{R}_+$ such that, after rescaling distances by $C_p/n^{1/4}$, M_n converges in distribution for the Gromov-Hausdorff-Prokhorov topology towards the Brownian map. To prove the preceding fact, we introduce a bootstrapping principle for distributional convergence of random labelled plane trees. In particular, the latter allows to obtain an invariance principle for labeled multitype Galton-Watson trees, with only a weak assumption on the centering of label displacements.

1. INTRODUCTION

1.1. Convergence of random planar maps. A *planar map* is an embedding of a finite connected graph into the two-dimensional sphere, viewed up to orientation-preserving homeomorphisms. For $p \geq 3$, a p -angulation is a planar map whose faces all have degree p . Scaling limits of random planar maps have been the subject of a lot of attention in recent years; perhaps the most celebrated results are the independent proofs by Miermont [20] and Le Gall [12] of the fact that the scaling limit of random 4-angulations (or quadrangulations) is the Brownian map. In fact, in his work Le Gall also established that, for $p = 3$ or $p \geq 4$ even, the scaling limit of p -angulations is the Brownian map. The current paper establishes the analogous result for p -angulations with $p \geq 5$ odd.

Theorem 1. *Let $p \geq 5$ be an odd integer and let (\mathbf{M}_n) be a sequence of independent random maps, such that for any $n \geq 1$, \mathbf{M}_n is a uniform p -angulation with n vertices. Denote by $\text{dist}_{\mathbf{M}_n}$ the graph distance on \mathbf{M}_n and μ_n the uniform probability distribution on its set of vertices $V(\mathbf{M}_n)$. Then there exists a constant C_p such that, as n goes to infinity,*

$$\left(V(\mathbf{M}_n), \frac{C_p}{n^{1/4}} \text{dist}_{\mathbf{M}_n}, \mu_n \right) \xrightarrow{d} (M, d^*, \lambda),$$

for the Gromov-Hausdorff-Prokhorov topology and where (M, d^*, λ) is the Brownian map.

We obtain Theorem 1 as a particular case of a more general result stated in Theorem 14, which deals with the convergence to the Brownian map of so-called *regular critical Boltzmann maps*. Before giving further context for our result, and the ideas of its proof, let us first emphasize that it relies on the work of Miermont and Le Gall and does not constitute an independent proof of the uniqueness of the limiting object.

The main motivation for our work is the conjecture that the Brownian map is a universal limiting object for many families of planar maps. As already mentioned, the Brownian map is known to be the scaling limit for uniform p -angulations for $p \in \{3\} \cup 2\mathbb{N}$, but also for quadrangulations without vertices of degree one [4], for simple triangulations and quadrangulations [2], for general maps [5], for bipartite maps [1] and for bipartite maps with prescribed degree sequence [16]. In this sense, our result is an additional step towards the universality of the Brownian map. Moreover, recall that a *bipartite* map is a map whose vertices can be partitioned into two sets, say B and W such that all edges in the map have one extremity in B and one extremity in W . It is easy to see that a planar map is

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bipartite if and only if all its faces have even degree. With the notable exception of [5] (which does not control the degree of faces of the maps considered), all the results listed above deal with either bipartite maps or with triangulations.

As with most results in this field, our work relies on a bijection between planar maps and labeled multitype trees: the Bouttier-di Francesco-Guitter bijection [7] plays this role in our case. Thanks to the general approach developed in [12], the only new result needed to prove Theorem 1 is that the encoding functions of multitype labeled trees associated to p -angulations by that bijection converge to the Brownian snake. Numerous results about convergence of labeled trees already exist in the literature (see for instance [18], [13]). However, most of these results rely on the assumption that the variation of labels along an edge is centered (see Section 2.4 below), or that degrees are bounded and the trees only have one type; such assumptions do not hold in our setting. To describe how we circumvent this difficulty, we introduce some further notations and definitions.

1.2. Symmetrization of labeled trees. Let t be a rooted plane tree. For a vertex v of t we write $k_t(v)$ for the number of children of v in t (k stands for “kids”). In the following we identify the vertex set $V(t)$ with the set of words given by the Ulam-Harris encoding. In this encoding, nodes are labeled by elements of $\bigcup_{n \geq 0} \mathbb{N}^n$, where $\mathbb{N}^0 = \{\emptyset\}$ by convention. The root receives label \emptyset ; the children of node $v = v_1 v_2 \dots v_h \in \mathbb{N}^h$ receive labels $(vi, 1 \leq i \leq k_t(v))$ in the order given by the plane embedding.

A *rooted labeled plane tree* is a pair $t = (t, d)$ where t is a rooted plane tree and $d = (d(e), e \in E(t)) \in \mathbb{R}^{E(t)}$ give the *displacements* of labels along the edges $E(t)$ of t .

A tree t is a multitype tree if each node $v \in V(t)$ has a type $s(v) \in \mathbb{Q}$ (any other fixed countable type space would work equally well). We likewise define multitype labeled plane trees. We may view an un-typed tree as a typed tree by giving all nodes the same type, so hereafter all trees in the paper are considered to be multitype, unless we mention otherwise.

Given a plane tree t , let $[t]$ be the set of plane trees which are isomorphic to t as multitype rooted trees (but not necessarily as multitype *plane* rooted trees). Write P_t for the set of vectors $\sigma = (\sigma_v, v \in V(t))$, where each σ_v is a permutation of $\{1, \dots, k_t(v)\}$. Such a vector σ uniquely specifies a tree $t' = \sigma(t) \in [t]$ as follows. Visually, reorder the children of each node v according to the permutation σ_v . Formally, for each node $v = v_1 v_2 \dots v_k \in V(t)$, there is a corresponding node $\sigma(v) \in V(t')$ whose type is the same as that of v and whose Ulam-Harris label is

$$\sigma(v) = \sigma_{\emptyset}(v_1) \sigma_{v_1}(v_2) \dots \sigma_{v_1 \dots v_{k-1}}(v_k).$$

If t is a labeled plane tree, we likewise define $[t]$ and $t' = \sigma(t) \in [t]$ by letting the labels follow their edges. Formally, if $e = uv \in e(t)$ then $d'(\sigma(u)\sigma(v)) = d(uv)$.

We typically use μ to denote a measure on unlabeled plane trees. We say such a measure μ is symmetric if $\mu(t) = \mu(t')$ for all plane trees t, t' with $t' \in [t]$. Similarly, ν will typically denote a measure on labelled plane trees, and we say such a measure ν is symmetric if $\nu((t, d)) = \nu((t', d'))$ whenever $(t', d') \in [(t, d)]$.

This work establishes a tool for establishing distributional convergence of random labelled plane trees, if such convergence is already known for symmetrized versions of the trees. We exclusively consider random labeled plane trees (T, D) satisfying the following three properties.

- (i) The law μ of the underlying unlabeled plane tree is symmetric: $\mu(\{t\}) = \mathbf{P}\{T = t\} = \mathbf{P}\{T = t'\} = \mu(\{t'\})$ whenever $t' \in [t]$.
- (ii) For each $v \in V(T)$, let $D_v = (D_{v,v_1}, \dots, D_{v,v_{k(v)}})$ be the vector of displacements from v to its children. Then the vectors $(D_v, v \in V(T))$ are conditionally independent given T .

- (iii) The law of each vector D_v is determined by the type of v together with the vector of types of its children.

If (T, D) satisfies all three properties then we say its law ν is *valid*.

Fix a plane tree t and let $\sigma = (\sigma_v, v \in V(t)) \in_u P_t$ be a uniformly random element of P_t . We call the random tree $\sigma(t)$ the *symmetrization* of t . This also makes sense for a random tree T ; in this case, conditionally given T , we have $\sigma \in_u P_T$, and the symmetrization is the tree $\sigma(T)$. If μ is the law of T then we write μ^{sym} for the law of its symmetrization. Note that μ is symmetric if and only if $\mu = \mu^{\text{sym}}$.

The definitions of the preceding paragraph all have analogues for labeled plane trees. The symmetrization of (t, d) , is $\sigma(t, d)$, where $\sigma \in_u P_t$; if ν is the law of random labeled plane tree (T, D) then we write ν^{sym} for the law of the symmetrization of (T, D) ; and, ν is symmetric if and only if $\nu = \nu^{\text{sym}}$. The following theorem establishes that certain asymptotic distributional properties of random labeled plane trees are unchanged by symmetrization of the distributions involved. The proof makes reference to the contour and label processes of trees; these are defined in Section 2, immediately following the statement of the theorem.

Theorem 2. *For each $n \geq 1$ let $T_n = (T_n, D_n)$ be a random labeled multitype plane tree whose law ν_n is valid, and let T_n^{sym} have law ν_n^{sym} . Write $(C_{T_n}(t), Z_{T_n}(t))_{0 \leq t \leq 1}$ and $(C_{T_n^{\text{sym}}}(t), Z_{T_n^{\text{sym}}}(t))_{0 \leq t \leq 1}$ for the contour and label processes of T_n and T_n^{sym} , respectively.*

Suppose there exist positive sequences $(a_n, n \geq 1)$ and $(b_n, n \geq 1)$ with $a_n \rightarrow 0$ and $b_n \rightarrow 0$, and a $C([0, 1], \mathbb{R}^2)$ -valued random process $(C, Z) = ((C(t), Z(t)), t \in [0, 1])$, such that

$$(a_n C_{T_n^{\text{sym}}}, b_n Z_{T_n^{\text{sym}}}) \xrightarrow{d} (C, Z)$$

for the uniform topology. Then, the following convergence also holds for the uniform topology:

$$(a_n C_{T_n}, b_n Z_{T_n}) \xrightarrow{d} (C, Z).$$

Let us conclude by coming back to planar maps. For $p \geq 5$ and $p \notin 2\mathbb{N}$, the push-forward of the uniform measure on p -angulations (or indeed of any regular critical Boltzmann distribution on maps) by the Bouttier-di Francesco-Guitter (discussed further in Section 4.3, below) is a probability distribution on labeled plane trees which is valid but not symmetric. Theorem 2 allows the transfer of results of Miermont [18] to establish that contour and label processes of these trees have the expected scaling limit (see Theorem 17 for a more precise statement).

2. VALID LAWS, FINITE-DIMENSIONAL-DISTRIBUTIONS AND SYMMETRIZATION

2.1. Definitions and notation. In this section t always denote a rooted plane multitype tree and \mathfrak{t} a labeled rooted plane multitype tree. For a vertex $v \in \mathfrak{t}$, the *child type vector* of v is the vector $\text{ctype}(v) = (s(v_i), 1 \leq i \leq k_t(v)) \in \mathbb{Q}^{k_t(v)}$.

We use the notation $|\mathfrak{t}| = |t| = |V(t)|$ interchangeably. Writing $n = 2|t| - 2$, we define the contour exploration $\theta = \theta_t : \{0, \dots, n\} \rightarrow V(t)$ of t inductively as follows. Let $\theta(0) = \emptyset$. Then, for $1 \leq i \leq n$, let $\theta(i)$ be the lexicographically first child of $\theta(i-1)$ that is not an element of $\{\theta(0), \dots, \theta(i-1)\}$, or let $\theta(i)$ be the parent of $\theta(i-1)$ if no such node exists.

For $v, w \in V(t)$, we write $\text{dist}(v, w) = \text{dist}_t(v, w)$ for the graph distance between v and w in t . We write $h(v) = \text{dist}_t(\emptyset, v)$ for the distance from v to the root, so if $v \in \mathbb{N}^k$ then $h(v) = k$. Note that $h(v)$ is the length of v in the Ulam-Harris encoding. The label of a vertex v of $\mathfrak{t} = (t, d)$, denoted $\ell_t(v)$ or $\ell(v)$, is defined as the sum of the edge labels on the path between v and the root. Formally, if $v = v_1 \dots v_k$ then $\ell_t(v) = d(\emptyset, v_1) + \dots + d(v_1 \dots v_{k-1}, v_1 \dots v_k)$.

The contour and label processes C_t and Z_t are the functions from $[0, 1]$ to \mathbb{R} defined as follows. For $0 \leq i \leq n$, set $C_t(i/n) = h(\theta(i))$ and $Z_t(i/n) = \ell(\theta(i))$. Then extend the domains of definition of C_t and Z_t to $[0, 1]$ by linear interpolation.

2.2. Notes on treelike paths. The metric structure of t may be recovered from $C = C_t$ as follows. For $x, y \in [0, 1]$ with $x \leq y$, let

$$\text{Dist}_C(x, y) = \text{Dist}_C(y, x) = C_t(x) + C_t(y) - 2 \inf\{C_t(u), x \leq u \leq y\}.$$

Then for all $0 \leq i, j \leq n$, $\text{dist}_t(\theta_t(i), \theta_t(j)) = \text{Dist}_C(i/n, j/n)$. In particular, if $\text{Dist}_C(i/n, j/n) = 0$ then $\theta_t(i) = \theta_t(j)$. Let \sim_C be the equivalence relation $\{(x, y) : \text{Dist}_C(x, y) = 0\}$, and let $\mathcal{T}_C = [0, 1]/\sim_C$. Then $(\mathcal{T}_C, \text{Dist}_C)$ is a metric space (here, by Dist_C we really mean its push-forward to \mathcal{T}_C), and its subspace induced by the points $\{[i/n], 0 \leq i \leq n\}$ is isometric to t .

Using the equivalence of dist_t and Dist_C at lattice times, if $\text{Dist}_C(i/n, j/n) = 0$ then $Z_t(i/n) = Z_t(j/n)$. Since Z_t is defined by linear interpolation, it follows that $Z_t(x) = Z_t(y)$ whenever $\text{Dist}_C(x, y) = 0$, and that the push-forward Z of Z_t to \mathcal{T}_C is well-defined and continuous for Dist_C . We then clearly have $Z([i/n]) = \ell_t(\theta_t(i))$ for all $0 \leq i \leq n$.

The two preceding paragraphs may be viewed as motivation for the following general construction. Say $\zeta \in C([0, 1], \mathbb{R}^+)$ is an excursion if $\zeta(0) = \zeta(1) = 0$. If ζ is an excursion then we may define Dist_ζ and \mathcal{T}_ζ just as above, and $(\mathcal{T}_\zeta, \text{Dist}_\zeta)$ is always a compact metric space. (In fact, it is always an \mathbb{R} -tree.)

Now fix a pair (ζ, f) with $\zeta \in C([0, 1], \mathbb{R}^+)$ and $f \in C([0, 1], \mathbb{R})$. We say (ζ, f) is a *tree-like path* if ζ is an excursion, $f(0) = f(1) = 0$, and for all $x, y \in [0, 1]$, if $\text{Dist}_\zeta(x, y) = 0$ then $f(x) = f(y)$. This implies that the push-forward of f to \mathcal{T}_ζ is well-defined and is continuous for Dist_ζ .

It follows from the results in [15] that if (C, Z) is a distributional limit as in Theorem 2 then (C, Z) is a tree-like path, a fact which will be useful in the proof of the theorem.

2.3. Valid distributions and labeled Galton-Watson trees. Fix a set $S \subseteq \mathbb{Q}$ of types and write S^{fin} for the set of finite vectors with entries from S ; we include the empty vector $()$ in this set. A multitype Galton-Watson tree with type space S is defined by a collection $p = (p^s, s \in S)$ with each p^s a probability distribution on S^{fin} , such that if $\vec{v}, \vec{w} \in S^{\text{fin}}$ are such that \vec{w} is a permutation of \vec{v} then $p^s(\vec{v}) = p^s(\vec{w})$ for all $s \in S$. Observe that, writing $n_x(\vec{v})$ for the number of entries of \vec{v} equal to $x \in S$, this means each p^s is uniquely determined by the values

$$p^s(\{\vec{v} \in S^{\text{fin}} : \forall x \in S, n_x(\vec{v}) = i_x\}),$$

as $(i_x, x \in S)$ ranges over collections of non-negative integers with finite sum.

A random tree \mathcal{T} is Galton-Watson(p)-distributed if the following holds. For all $n \geq 1$ and all rooted plane trees t with node types in S and with height at most n , writing s for the type of the root of t ,

$$\mathbf{P}\{\mathcal{T}_{\leq n} = t \mid s(\emptyset) = s\} = \prod_{v \in V(t_{<n})} p^{s(v)}(\text{ctype}_t(v)).$$

Here we have written $\mathcal{T}_{\leq n}$ for the subtree of \mathcal{T} consisting of nodes at distance at most n from the root (and likewise defined $t_{<n}$); also, we use the convention that $\text{ctype}_t(v) = ()$ if v has no children. In particular, this implies that, for t a finite tree,

$$\mathbf{P}\{\mathcal{T} = t \mid s(\emptyset) = s\} = \prod_{v \in V(t)} p^{s(v)}(\text{ctype}_t(v)).$$

From the assumption that the offspring distributions $(p^s, s \in S)$ are permutation-invariant, the following result is immediate.

Proposition 3. *Let \mathcal{T} be a multitype Galton-Watson tree with type space $S \subseteq \mathbb{Q}$. Fix any type $s \in S$ and integer $n \geq 1$ such that $\mathbf{P}\{s(\emptyset) = s, |\mathcal{T}| = n\} > 0$. Then the conditional probability measure $\mathbf{P}\{\cdot \mid s(\emptyset) = s, |\mathcal{T}| = n\}$ is valid.*

We next consider how the definition of valid laws interacts with that of multi-type Galton-Watson trees. Let (T, D) be a random labeled tree with type space $S \subseteq \mathbb{Q}$. Then (T, D) satisfies (ii) and (iii) in the definition of validity if and only if there exists a set $\{\pi_s^r : r \in S, s \in \bigcup_{k \geq 1} S^k\}$, with each $\pi_{(s_1, \dots, s_k)}^r$ a probability measure on \mathbb{R}^k , such that for any plane tree t and any sets $(B_v, v \in V(t))$ with B_v a Borel subset of $\mathbb{R}^{k_t(v)}$ for all v ,

$$\mathbf{P}\{T = t, \forall v \in V(t), D_v \in B_v\} = \mathbf{P}\{T = t\} \cdot \prod_{v \in V(t)} \pi_{\text{ctype}(v)}^{s(v)}(B_v). \quad (1)$$

In other words, writing π_T for the conditional law of D given T , we have $\pi_T = \otimes_{v \in V(T)} \pi_{\text{ctype}(v)}^{s(v)}$.

Now suppose T is a Galton-Watson(p) tree conditioned on its size and on the type of its root; in other words, T has law $\mathbf{P}\{\cdot \mid s(\emptyset) = s, |\mathcal{T}| = n\}$ where \mathcal{T} is Galton-Watson(p)-distributed and $s \in S$. Then condition (i) in the definition of validity is automatic, and (1) may be re-expressed as

$$\mathbf{P}\{T = t, \forall v \in V(t), D_v \in B_v\} = \prod_{v \in V(t)} p^{s(v)}(\text{ctype}(v)) \cdot \pi_{\text{ctype}(v)}^{s(v)}(B_v).$$

2.4. Displacement locally centered, centered and globally centered. Let (T, D) be a tree sampled from a valid distribution. We consider $\boldsymbol{\pi} = \{\pi_s^r : r \in S, s \in \bigcup_{k \geq 1} S^k\}$, the family of the distributions of the vector of displacements.

Definition 4. *For each $\pi_s^r \in \boldsymbol{\pi}$, let $(X_{s,i}^r, 1 \leq i \leq |s|)$ have law π_s^r . We say $\boldsymbol{\pi}$ is locally centered if $\mathbf{E}[X_{s,i}^r]$ for all $r \in \mathbb{Q}$, $s \in \bigcup_{k \geq 1} \mathbb{Q}^k$ and $i \in \{1, \dots, |s|\}$.*

The family $\boldsymbol{\pi}$ is centered if for any $r \in \mathbb{Q}$ and any $z \in \mathbb{Z}_+^{\mathbb{Q}}$,

$$\sum_{\substack{s \in \mathbb{Q}^k \\ p(s)=z}} \sum_{i=1}^k \mathbf{E}[X_{s,i}^r] = 0.$$

The assumption that $\boldsymbol{\pi}$ is locally centered appears often in the litterature about the convergence of labeled Galton-Watson trees. As already mentioned in the introduction, this assumption does not hold for the family of trees we want to study. However, our trees are centered. The following claim should be clear from the definition of symmetrization.

Claim 5. *If (T, D) is a tree sampled from a valid distribution with centered displacements, then the family of displacements distributions of $(T^{\text{sym}}, D^{\text{sym}})$ is locally centered.*

Note that in [13], Marckert studies the convergence to the Brownian snake for labelled single type Galton-Watson trees, where the offspring distribution ζ is assumed to have bounded support and where the displacements are assumed to be globally centered, in that

$$\sum_{k \geq 0} \zeta(k) \sum_{i=1}^k \mathbf{E}[X_{k,i}] = 0,$$

where $(X_{k,1}, \dots, X_{k,k})$ has the distribution of vector displacements to the children for a node with k children. We could not find a way to use symmetrization to transform globally centered trees into locally centered trees. It is an open problem to know whether the bounded support assumption in Marckert's work can be relaxed.

2.5. Subsampling in labeled trees. A important step of the proof of Theorem 2 is accomplished by the following lemma, which relates the laws of subtrees obtained by sampling in random labeled trees and their symmetrizations. Informally, the lemma states that the distribution of the subtree spanned by a set of randomly sampled vertices is the same in a random labeled tree and in its symmetrization, provided that all labels on child edges incident to branchpoints of the subsampled trees are ignored.

For a plane tree t and a sequence $\mathbf{v} = (v_1, \dots, v_k) \in V(t)^k$, write $t(\mathbf{v})$ for the subtree of t spanned by the vertices of \mathbf{v} together with all their ancestors in t . We view $t(\mathbf{v})$ as a plane tree by using the plane structure of t . The Ulam-Harris labels in $t(\mathbf{v})$ need not agree with those in t , so for v a weak ancestor of a vertex in \mathbf{v} , we write $U(\mathbf{v}, v)$ for (the Ulam-Harris label of) the node corresponding to v in $t(\mathbf{v})$. We also let $U(\mathbf{v}) = (U(\mathbf{v}, v_1), \dots, U(\mathbf{v}, v_k))$; this vector plays a key role in the coming lemma.

If $\mathbf{d} \in \mathbb{R}^{e(t)}$ is a labeling of t then we let $\mathbf{d}(\mathbf{v})$ be the pushforward of \mathbf{d} to $t(\mathbf{v})$; so if $e = uu' \in e(t(\mathbf{v}))$ with $u = U(\mathbf{v}, v)$, $u' = U(\mathbf{v}, v')$ then $\mathbf{d}(e)_{uu'} = \mathbf{d}_{vv'}$. We also define a modified labeling $\mathbf{d}\langle\mathbf{v}\rangle$ as follows. For an edge $uu' \in e(t(\mathbf{v}))$ with u the parent of u' , let

$$\mathbf{d}\langle\mathbf{v}\rangle_{uu'} = \begin{cases} \mathbf{d}(\mathbf{v})_{uu'} & \text{if } k_{t(\mathbf{v})}(u) = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Think of $\mathbf{d}\langle\mathbf{v}\rangle$ as “ignoring displacements at branchpoints of $t(\mathbf{v})$ ”.

Let $\mathbf{T} = (T, D)$ be a random labeled plane tree. We say that a random vector $\mathbf{R} = (R_1, \dots, R_k)$ is uniformly sampled from \mathbf{T} if for all plane trees t , Borel sets $(B_e, e \in e(t))$ and vectors $\mathbf{r} = (r_1, \dots, r_k) \in V(t)^k$,

$$\mathbf{P}\{T = t, \mathbf{R} = \mathbf{r}, D_e \in B_e \forall e \in e(t)\} = \frac{1}{|V(t)|^k} \mathbf{P}\{T = t, D_e \in B_e \forall e \in e(t)\}.$$

Lemma 6. *Let $\mathbf{T} = (T, D)$ be a random labeled multitype plane tree with valid law ν , and let $\mathbf{T}^{\text{sym}} = (T^{\text{sym}}, D^{\text{sym}})$ have law ν^{sym} .*

Fix $k \in \mathbb{N}$, and let $\mathbf{R} = (R_1, \dots, R_k)$ and $\mathbf{Q} = (Q_1, \dots, Q_k)$ be random vectors of length k , uniformly sampled from \mathbf{T} and \mathbf{T}^{sym} respectively. Then $(T(\mathbf{R}), D\langle\mathbf{R}\rangle, U(\mathbf{R}))$ and $(T^{\text{sym}}(\mathbf{Q}), D^{\text{sym}}\langle\mathbf{Q}\rangle, U(\mathbf{Q}))$ are equal in distribution.

Proof. If ν is valid and the displacement laws under ν are described by measures $\{\pi_s^r : r \in \mathbb{Q}, s \in \bigcup_{k \geq 1} \mathbb{Q}^k\}$, then ν^{sym} is also valid and the displacement laws under ν are described by measures $\{\pi_s^{r, \text{sym}} : r \in \mathbb{Q}, s \in \bigcup_{k \geq 1} \mathbb{Q}^k\}$.

We warm up by proving the lemma in the case of single type trees. In this case we may write π_k for the unique displacement law for nodes with k children, and the measure π_k^{sym} is given by

$$\pi_k^{\text{sym}}(B) = \frac{1}{k!} \sum_{\rho \in \mathfrak{S}_k} \pi_k(\rho(B)),$$

for $B \subset \mathbb{R}^k$ Borel, where $\rho(x_1, \dots, x_k) := (x_{\rho^{-1}(1)}, \dots, x_{\rho^{-1}(k)})$ and $\rho(B) = \{\rho(x), x \in B\}$.

Given a tree t and a sequence $\mathbf{u} = (u_1, \dots, u_k) \in V(t)^k$, let $\text{branch}(t, \mathbf{u})$ be the set of vertices of t possessing at least two distinct children with descendants in \mathbf{u} (in other words, these are the vertices of t which correspond to branchpoints of $t(\mathbf{u})$). Further, let $P_{(t, \mathbf{u})}$ be the set of vectors $\sigma = (\sigma_v, v \in t) \in P_t$ such that σ_v is the identity permutation for all $v \in \text{branch}(t, \mathbf{u})$.

Now let (T, D) have law ν , let $\mathbf{R} = (R_1, \dots, R_k) \in V(T)^k$ be a random vector of length k uniformly sampled from T , and let $\sigma \in_u P_{(T, \mathbf{R})}$. We construct another labelled tree (\hat{T}, \hat{D}) from (T, D) as follows. In words we use σ to perform a full symmetrization at all vertices of T except at vertices corresponding the branchpoints of $T(\mathbf{R})$; at the latter vertices we don't permute the children and we set all the displacements to zero. Formally, set $\hat{T} = \sigma(T)$,

for $1 \leq i \leq k$ set $\hat{R}_i = \sigma(R_i)$, and let $\hat{R} = (\hat{R}_1, \dots, \hat{R}_k)$. Then for $(v, vi) \in e(T)$ let

$$\hat{D}_{\sigma(v), \sigma(vi)} = \begin{cases} 0 & \text{if } v \in \text{branch}(T, R) \\ D_{v, vi} & \text{otherwise,} \end{cases}$$

so if $e = uv \in E(\hat{T})$ then $\hat{D}_e = D_{\sigma^{-1}(v)\sigma^{-1}(u)}$. Now set $\hat{D} = (\hat{D}_e, e \in e(\hat{T}))$.

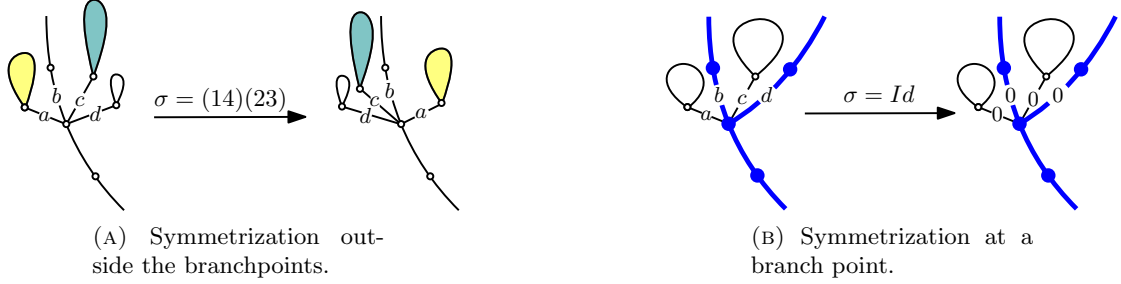


FIGURE 1. Examples of symmetrization. In both figures, the tree t is on the left and \hat{t} on the right. Labels a, b, c and d represent the displacements along the edges. Branches of $t(u)$ are represented in bold blue.

Here is an important property of the preceding construction. For all $1 \leq i, j \leq k$, if $R_i \prec R_j$ (where \prec corresponds to the lexicographic ordering on the Ulam-Harris encoding) then $\hat{R}_i \prec \hat{R}_j$. This immediately implies that $(T(R), U(R)) = (\hat{T}(\hat{R}), U(\hat{R}))$. Moreover, the only differences between $(T(R), D(R))$ and $(\hat{T}(\hat{R}), \hat{D}(\hat{R}))$ occur at nodes with at least two children whose subtrees contain elements of R . Since the displacements on edges leaving such nodes are set to 0 when passing from $D(R)$ to $D(\hat{R})$ (see (2)), it follows that $\hat{D}(\hat{R}) = D(\hat{R})$ as well, so $(\hat{T}(\hat{R}), \hat{D}(\hat{R}), U(\hat{R})) = (T(R), D(R), U(R))$.

For $v \in V(\hat{T})$ we write $\hat{D}_v = (\hat{D}_{v, vi}, 1 \leq i \leq k(v))$. Now fix a tree t and a length- k vector $r = (r_1, \dots, r_k)$ of vertices of t . We will show that for any collection $(B_v, v \in V(t) \setminus \text{branch}(t, r))$, with each B_v a Borel set of $\mathbb{R}^{k_{\iota}(v)}$,

$$\begin{aligned} & \mathbf{P} \left\{ (\hat{T}, \hat{R}) = (t, r) \text{ and } \hat{D}_v \in B_v \forall v \in V(t) \setminus \text{branch}(t, r) \right\} \\ &= \mathbf{P} \left\{ (T^{\text{sym}}, Q) = (t, r) \text{ and } D_v^{\text{sym}} \in B_v \forall v \in V(t) \setminus \text{branch}(t, r) \right\}. \end{aligned}$$

This equality implies that $(\hat{T}(\hat{R}), \hat{D}(\hat{R}), U(\hat{R}))$ and $(T^{\text{sym}}(Q), D^{\text{sym}}(Q), U(Q))$ are equal in distribution. Since $(\hat{T}(\hat{R}), \hat{D}(\hat{R}), U(\hat{R})) = (T(R), D(R), U(R))$, this will prove the lemma for single type trees.

For $\tau \in P_{(t, r)}$, we let $\tau(t, r) = (\tau(t), (\tau(r_1), \dots, \tau(r_k)))$. For $\tau \in P_t$, we let τ^* be the element of $P_{\tau(t)}$ defined by setting $\tau_{\tau(v)}^* = (\tau_v)^{-1}$ for all $v \in t$. Note that $\tau^*(\tau(t)) = t$ – so τ^* acts as an inverse to τ – and that if $\tau \in P_{(t, r)}$ then $\tau^* \in P_{\tau(t, r)}$.

Since $\hat{T} = \sigma(T)$ we have

$$\begin{aligned} & \mathbf{P} \left\{ (\hat{T}, \hat{R}) = (t, r) \text{ and } \hat{D}_v \in B_v \forall v \in V(t) \setminus \text{branch}(t, r) \right\} \\ &= \sum_{\tau \in P_{(t, r)}} \mathbf{P} \left\{ \tau^*(T, R) = (t, r), \sigma = \tau^* \text{ and } \hat{D}_v \in B_v \forall v \in V(t) \setminus \text{branch}(t, r) \right\} \\ &= \sum_{\tau \in P_{(t, r)}} \mathbf{P} \left\{ (T, R) = \tau(t, r), \sigma = \tau^* \text{ and } D_{\tau(v)} \in \tau_v(B_v) \forall v \in V(t) \setminus \text{branch}(t, r) \right\}, \end{aligned}$$

Now note that

$$|P_{(t,r)}| = \prod_{v \in V(t) \setminus \text{branch}(t,r)} k_t(v)! = |P_{\tau(t,r)}|,$$

as $\text{branch}(\tau(t,r)) = \{\tau(v), v \in \text{branch}(t,r)\}$. Using that T is symmetric, that the elements of \mathbb{R} are uniformly sampled from T , and that $\sigma \in_u P_{(T,\mathbb{R})}$, it follows that for all $\tau \in P_t$,

$$\mathbf{P}\{(T, \mathbb{R}) = \tau(t,r), \sigma = \tau^*\} = \frac{\mathbf{P}\{T = t\}}{|t|^k} \cdot \frac{1}{|P_{\tau(t,r)}|} = \frac{\mathbf{P}\{T = t\}}{|t|^k} \cdot \frac{1}{|P_{(t,r)}|}. \quad (3)$$

We note for later use that, thus far, we have not used that T is a single-type tree, so (3) also holds in the multitype setting.

Recalling the notation π_k and π_k^{sym} from the start of the lemma, and using that $|P_{(t,r)}| = \prod_{v \in V(t) \setminus \text{branch}(t,r)} k_t(v)!$ as well as (1), it follows that

$$\begin{aligned} & \mathbf{P}\left\{(\hat{T}, \hat{\mathbb{R}}) = (t,r) \text{ and } \hat{D}_v \in B_v \forall v \in V(t) \setminus \text{branch}(t,r)\right\} \\ &= \frac{\mathbf{P}\{T = t\}}{|t|^k} \cdot \sum_{\tau \in P_{(t,r)}} \frac{1}{|P_{(t,r)}|} \prod_{v \in V(t) \setminus \text{branch}(t,r)} \pi_{k_t(v)}(\tau_v(B_v)) \\ &= \frac{\mathbf{P}\{T = t\}}{|t|^k} \cdot \prod_{v \in V(t) \setminus \text{branch}(t,r)} \left(\frac{1}{k_t(v)!} \sum_{\rho \in \mathfrak{S}_{k_t(v)}} \pi_{k_t(v)}(\rho(B_v)) \right) \\ &= \frac{\mathbf{P}\{T = t\}}{|t|^k} \cdot \prod_{v \in V(t) \setminus \text{branch}(t,r)} \pi_{k_t(v)}^{\text{sym}}(B_v) \\ &= \mathbf{P}\{(T^{\text{sym}}, \mathbb{Q}) = (t,r) \text{ and } D_v^{\text{sym}} \in B_v \forall v \in t \setminus \text{branch}(t,r)\}, \end{aligned}$$

as required, which completes the proof for the monotype case. The key point in the argument is that, because we ignore displacements at branchpoints, the probability factorizes due to (1).

We now move to the multitype case. We start by remarking that if ν is valid and the displacement laws under ν are described by measures $\{\pi_s^u : u \in \mathbb{Q}, s \in \bigcup_{k \geq 1} \mathbb{Q}^k\}$, then ν^{sym} is also valid and the displacement laws under ν are described by the measures $\{\pi_s^{u,\text{sym}} : u \in \mathbb{Q}, s \in \bigcup_{k \geq 1} \mathbb{Q}^k\}$. The definition of symmetrization implies that, for any type u and any $k \geq 1$ any type vector $s \in \mathbb{Q}^k$ and any $B \subset \mathbb{R}^k$ Borel,

$$\pi_s^{u,\text{sym}}(B) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \pi_{\sigma(s)}^u(\sigma(B)).$$

Now note that for any $v \in V(t)$ and any $\tau \in P_{(\tau,r)}$, $s(\tau(v)) = s(v)$ and $\text{ctype}(\tau(v)) = \tau_v(\text{ctype}(v))$. Together with the preceding identity and (1) and (3), this implies that

$$\begin{aligned} & \mathbf{P}\left\{(\hat{T}, \hat{\mathbb{R}}) = (t,r) \text{ and } \hat{D}_v \in B_v \forall v \in V(t) \setminus \text{branch}(t,r)\right\} \\ &= \frac{\mathbf{P}\{T = t\}}{|t|^k} \cdot \sum_{\tau \in P_{(t,r)}} \frac{1}{|P_{(t,r)}|} \prod_{v \in V(t) \setminus \text{branch}(t,r)} \pi_{\text{ctype}(\tau(v))}^{s(\tau(v))}(\tau_v(B_v)) \\ &= \frac{\mathbf{P}\{T = t\}}{|t|^k} \cdot \sum_{\tau \in P_{(t,r)}} \frac{1}{|P_{(t,r)}|} \prod_{v \in V(t) \setminus \text{branch}(t,r)} \pi_{\tau_v(\text{ctype}(v))}^{s(v)}(\tau_v(B_v)) \\ &= \frac{\mathbf{P}\{T = t\}}{|t|^k} \cdot \prod_{v \in V(t) \setminus \text{branch}(t,r)} \pi_{\text{ctype}(v)}^{s(v),\text{sym}}(B_v) \\ &= \mathbf{P}\{(T^{\text{sym}}, \mathbb{Q}) = (t,r) \text{ and } D_v^{\text{sym}} \in B_v \forall v \in t \setminus \text{branch}(t,r)\}, \end{aligned}$$

which concludes the proof of the lemma. \square

3. PROOF OF THEOREM 2

In this section we explain how Theorem 2 follows from Lemma 6. For the remainder of the section, fix labeled trees $T_n = (T_n, D_n)$ and $T_n^{\text{sym}} = (T_n^{\text{sym}}, D_n^{\text{sym}})$ for $n \geq 1$, satisfying the conditions in Theorem 2, and suppose that there exists a random $C([0, 1], \mathbb{R}^2)$ -valued random process (C, Z) and positive sequences $(a_n, n \geq 1)$ and $(b_n, n \geq 1)$ with $a_n \rightarrow 0$ and $b_n \rightarrow 0$ such that

$$(a_n C_{T_n^{\text{sym}}}, b_n Z_{T_n^{\text{sym}}}) \xrightarrow{d} (C, Z)$$

for the uniform topology. We must show that the same distributional limit obtains for $(a_n C_{T_n}, b_n Z_{T_n})$. In the coming arguments we write $\theta_n = \theta_n$, $C_n = C_{T_n}$, $Z_n = Z_{T_n}$ and $\ell_n = \ell_{T_n}$ to simplify notation, and similarly write $\text{Dist}_{T_n^{\text{sym}}}^{\text{sym}} = \text{Dist}_{C_{T_n^{\text{sym}}}}$ et cetera.

In brief, the proof proceeds as follows. To prove convergence in distribution it suffices to prove convergence of finite-dimensional distributions (FDDs), plus tightness. Lemma 6 will yield convergence of random FDDs; by sampling sufficiently many random points we may use this to show convergence of arbitrary FDDs. Tightness will follow fairly easily from the convergence for the symmetrized process and fact that, aside from the plane structure, a labeled tree is identical to its symmetrization.

In the proof of the next lemma we use the following definition. Fix a plane tree t and write $n = 2|V(t)| - 2$. For $0 \leq y < 1$, let $v(t, y)$ be whichever of $\theta_t(\lfloor ny \rfloor)$ and $\theta_t(\lfloor ny \rfloor + 1)$ is further from the root. Note that if U is uniformly distributed on $[0, 1]$ then $v(t, U)$ is a uniformly random non-root node of t .

Lemma 7. *Let $(X_i, i \geq 1)$ be independent Uniform $[0, 1]$ random variables, independent of the trees T_n . Fix $k \geq 1$ and write $(X_i^\uparrow, 1 \leq i \leq k)$ for the increasing reordering of X_1, \dots, X_k . Then*

$$(a_n C_{T_n}(X_i^\uparrow), b_n Z_{T_n}(X_i^\uparrow), i \leq k) \xrightarrow{d} (C(X_i^\uparrow), Z(X_i^\uparrow), i \leq k)$$

as $n \rightarrow \infty$.

Proof. Fix $n \geq 1$ and $k \geq 1$, let $T = T_n$ and $T^{\text{sym}} = T_n^{\text{sym}}$, and then let $R = (R_1, \dots, R_k)$ and $Q = (Q_1, \dots, Q_k)$ be vectors of independent uniform samples from T and T^{sym} as in Lemma 6. The conclusion of that lemma implies that we may find a coupling under which

$$(T(R), D\langle R \rangle, U(R)) \stackrel{\text{a.s.}}{=} (T^{\text{sym}}(Q), D^{\text{sym}}\langle Q \rangle, U(Q)).$$

For each $1 \leq i \leq k$, let $R'_i = v(T_n, X_i)$ and let $Q'_i = v(T_n^{\text{sym}}, X_i)$; then let $R' = (R'_i, i \leq k)$ and let $Q' = (Q'_i, i \leq k)$. Let E be the event that $\emptyset \notin \{R_1, \dots, R_k\}$, so the conditional law of $(T(R), U(R))$ given E is the law of $(T_n(R'), U(R'))$. Since $Q \stackrel{\text{a.s.}}{=} R$, the event E is equivalent to the event that $\emptyset \notin \{Q_1, \dots, Q_k\}$. Thus the conditional law of $(Q, T^{\text{sym}}(Q))$ given E is the law of $(Q', T_n^{\text{sym}}(Q'))$. It follows that $(T_n(R'), D\langle R' \rangle, U(R'))$ and $(T_n^{\text{sym}}(Q'), D^{\text{sym}}\langle Q' \rangle, U(Q'))$ are identically distributed.

We may now fix a coupling which makes

$$(T_n(R'), D\langle R' \rangle, U(R')) \stackrel{\text{a.s.}}{=} (T_n^{\text{sym}}(Q'), D^{\text{sym}}\langle Q' \rangle, U(Q')).$$

Letting $R_i^\uparrow = v(T_n, X_i^\uparrow)$, $Q_i^\uparrow = v(T_n^{\text{sym}}, X_i^\uparrow)$ and $R^\uparrow = (R_i^\uparrow, i \leq k)$, $Q^\uparrow = (Q_i^\uparrow, i \leq k)$, we then have

$$(T_n(R^\uparrow), D\langle R^\uparrow \rangle, U(R^\uparrow)) \stackrel{\text{a.s.}}{=} (T_n^{\text{sym}}(Q^\uparrow), D^{\text{sym}}\langle Q^\uparrow \rangle, U(Q^\uparrow)). \quad (4)$$

Now write Δ_n and Δ_n^{sym} for the greatest absolute values of an edge label of T_n and T_n^{sym} , respectively. Then

$$\Delta_n = \sup_{uv \in e(T)} |\ell_n(u) - \ell_n(v)| \quad \text{and} \quad \Delta_n^{\text{sym}} = \sup_{uv \in e(T^{\text{sym}})} |\ell_n^{\text{sym}}(u) - \ell_n^{\text{sym}}(v)|.$$

For all $i \leq k$, the difference between $\ell_n(R_i^\uparrow)$ and the label of $U(R^\uparrow, R_i^\uparrow)$ in $(T_n(R^\uparrow), D\langle R^\uparrow \rangle)$ is at most $(k-1)\Delta_n$, since any difference between these labels is caused exclusively by the

zeroing of labels in $D\langle R^\uparrow \rangle$ at branchpoints, and there are at most $k - 1$ branchpoints of $T\langle R^\uparrow \rangle$ along any path from the root. Likewise, the difference between $\ell_n^{\text{sym}}(Q_i^\uparrow)$ and the label of $U(Q^\uparrow, Q_i^\uparrow)$ in $(T_n^{\text{sym}}(Q^\uparrow), D\langle R^\uparrow \rangle)$ is at most $(k - 1)\Delta_n^{\text{sym}}$. It then follows from (4) that

$$|\ell_n^{\text{sym}}(Q_i^\uparrow) - \ell_n(R_i^\uparrow)| \leq (k - 1)(\Delta_n + \Delta_n^{\text{sym}}).$$

Also, the values $Z_n(X_i^\uparrow)$ and $\ell_n(R_i^\uparrow)$ are the labels of nodes of T_n at distance at most one, and the values $Z_n^{\text{sym}}(X_i^\uparrow)$ and $\ell_n^{\text{sym}}(Q_i^\uparrow)$ are the labels of nodes of T_n^{sym} at distance at most one, so

$$|Z_n(X_i^\uparrow) - \ell_n(R_i^\uparrow)| \leq \Delta_n \text{ and } |Z_n^{\text{sym}}(X_i^\uparrow) - \ell_n^{\text{sym}}(Q_i^\uparrow)| \leq \Delta_n^{\text{sym}}.$$

It follows that

$$b_n \sup_{i \leq k} |Z_n^{\text{sym}}(X_i^\uparrow) - Z_n(X_i^\uparrow)| \leq kb_n(\Delta_n + \Delta_n^{\text{sym}}) \quad (5)$$

Now notice that, writing $N = 2|T^{\text{sym}}| - 2$, we may represent Δ_n and Δ_n^{sym} as

$$\Delta_n = \sup_{|x-y| \leq 1/N} |Z_n(x) - Z_n(y)| \quad \text{and} \quad \Delta_n^{\text{sym}} = \sup_{|x-y| \leq 1/N} |Z_n^{\text{sym}}(x) - Z_n^{\text{sym}}(y)|.$$

Since (C, Z) is a $C([0, 1], \mathbb{R}^2)$ -valued process, Z itself is a $C([0, 1], \mathbb{R})$ -valued process, so is almost surely uniformly continuous. Since $(a_n C_n^{\text{sym}}, b_n Z_n^{\text{sym}}) \xrightarrow{d} (C, Z)$ it follows that $b_n Z_n^{\text{sym}} \xrightarrow{d} Z$ for the uniform topology on $C([0, 1], \mathbb{R})$. The second of the preceding equalities then implies that $b_n \Delta_n^{\text{sym}} \xrightarrow{d} 0$.

For any labelled tree $t = (t, d)$ and any $\sigma \in P_t$, the multiset of edge labels is the same in t and in $\sigma(t)$, so in particular the largest absolute value of an edge label is the same in both trees. It thus follows from the definition of symmetrization that $\Delta_n \stackrel{d}{=} \Delta_n^{\text{sym}}$. Since $\Delta_n^{\text{sym}} \xrightarrow{d} 0$ it follows that $b_n \Delta_n \xrightarrow{d} 0$ as well, and (5) then implies that

$$b_n \sup_{i \leq k} |Z_n^{\text{sym}}(X_i^\uparrow) - Z_n(X_i^\uparrow)| \xrightarrow{d} 0.$$

Under the coupling which yields (4), we also have

$$(C_n(X_i^\uparrow), i \leq k) = (C_n^{\text{sym}}(X_i^\uparrow), i \leq k),$$

from which it follows that $(a_n C_n(X_i^\uparrow), b_n Z_n(X_i^\uparrow))$ and $(a_n C_n^{\text{sym}}(X_i^\uparrow), b_n Z_n^{\text{sym}}(X_i^\uparrow))$ must have the same distributional limit. By the assumption of Theorem 2, the latter converges to $(C(X_i^\uparrow), Z(X_i^\uparrow), i \leq k)$. \square

Lemma 8. *Let $T = (T, D)$ be a random labeled tree and let T^{sym} be its symmetrization. Then for any constants k, K, M , we have*

$$\begin{aligned} & \mathbf{P} \left\{ \sup_{|i-j| \leq k} |\ell_T(\theta_T(i)) - \ell_T(\theta_T(j))| > M \right\} \\ & \leq \mathbf{P} \left\{ \sup_{|i-j| \leq k} \text{dist}(\theta_T(i), \theta_T(j)) > K \right\} + \mathbf{P} \left\{ \sup_{v, w \in T^{\text{sym}}: \text{dist}(v, w) \leq K} |\ell_{T^{\text{sym}}}(v) - \ell_{T^{\text{sym}}}(w)| > M \right\}. \end{aligned}$$

Proof. For vertices v, w of a rooted plane tree, write $\llbracket v, w \rrbracket$ for the unique path between v and w ; this path is determined by the Ulam-Harris labels of v and w themselves, so there is no need to indicate the tree to which v and w belong in the notation. Now fix a labeled tree $t = (t, d)$ and let $\sigma \in P_t$. Note that for any $v, w \in V(t)$, the paths $\llbracket v, w \rrbracket$

and $\llbracket \sigma(v), \sigma(w) \rrbracket$ are identical, in that they have the same length, and visit edges with the same labels, in the same order. It follows that for all $i, j \leq 2|V(t)| - 2$,

$$\begin{aligned} \text{dist}(\theta(i), \theta(j)) &= \text{dist}(\sigma(\theta(i)), \sigma(\theta(j))) \text{ and} \\ \ell_t(\theta(i)) &= \ell_{\sigma(t)}(\sigma(\theta(i))). \end{aligned}$$

The second identity implies that for any k ,

$$\sup_{|i-j| \leq k} |\ell_t(\theta(i)) - \ell_t(\theta(j))| = \sup_{|i-j| \leq k} |\ell_{\sigma(t)}(\sigma(\theta(i))) - \ell_{\sigma(t)}(\sigma(\theta(j)))|.$$

The first identity implies that

$$\sup_{|i-j| \leq k} \text{dist}(\theta(i), \theta(j)) = \sup_{|i-j| \leq k} \text{dist}(\sigma(\theta(i)), \sigma(\theta(j))).$$

For any constants M, K , it follows that if

$$\sup_{|i-j| \leq k} |\ell_t(\theta(i)) - \ell_t(\theta(j))| > M$$

then either

$$\sup_{|i-j| \leq k} \text{dist}(\theta(i), \theta(j)) > K$$

or

$$\sup_{v, w \in V(\sigma(t)): \text{dist}(v, w) \leq K} |\ell_{\sigma(t)}(v) - \ell_{\sigma(t)}(w)| > M.$$

We now apply this to the random tree $\mathbb{T} = (T, D)$, with $\sigma \in_u P_{\mathbb{T}}$. Since $\sigma(\mathbb{T}) \stackrel{d}{=} \mathbb{T}^{\text{sym}}$, the result follows. \square

Lemma 9. *Under the hypotheses of Theorem 2, for all $\beta > 0$, there exists $\alpha = \alpha(\beta) > 0$ such that*

$$\limsup_n \mathbf{P} \left\{ \sup_{|i-j| \leq \lfloor \alpha |T_n| \rfloor} a_n \text{dist}(\theta_n(i), \theta_n(j)) > \beta \right\} < \beta.$$

Proof. Fix $\beta > 0$. Since μ_n is symmetric, T_n and T_n^{sym} have the same distribution (as unlabeled plane trees). Thus, the convergence result for the contour of (T_n^{sym}) translates immediately into the same result for the contour of (T_n) . This implies in particular that the process $(a_n C_n)$ is tight, so there exists $\alpha > 0$ such that

$$\limsup_n \mathbf{P} \left\{ \sup_{|x-y| \leq \alpha} a_n |C_n(x) - C_n(y)| > \beta \right\} < \beta. \quad (6)$$

Now, for any n and $0 \leq i \leq j \leq 2|T_n| - 2$, observe that

$$\text{dist}(\theta_n(i), \theta_n(j)) = C_n\left(\frac{i}{2|T_n| - 2}\right) + C_n\left(\frac{j}{2|T_n| - 2}\right) - 2 \inf_{i \leq k \leq j} C_n\left(\frac{k}{2|T_n| - 2}\right),$$

which together with (6) and the triangle inequality gives the desired result. \square

Lemma 10. *Under hypotheses of Theorem 2, for all $\epsilon > 0$, there exists $\beta = \beta(\epsilon) > 0$ such that*

$$\sup_n \mathbf{P} \left\{ \sup_{v, w \in T_n^{\text{sym}}: \text{dist}(v, w) \leq \beta/a_n} b_n |\ell_n^{\text{sym}}(v) - \ell_n^{\text{sym}}(w)| > \epsilon \right\} < \epsilon$$

Proof. As noted in the introduction, if (ζ, f) is a tree-like path then f can be pushed forward to \mathcal{T}_{ζ} and is continuous on that domain. Since \mathcal{T}_{ζ} is compact, f is in fact uniformly continuous on \mathcal{T}_{ζ} . In the current setting, this implies that the push-forward of Z to \mathcal{T}_C is

a.s. uniformly continuous on \mathcal{T}_C with respect to (the pushforward of) Dist_C . Thus, for all $\epsilon > 0$, there exists $\beta > 0$ such that

$$\mathbf{P} \left\{ \sup_{x,y \in [0,1]: \text{Dist}_C(x,y) \leq \beta} |Z(x) - Z(y)| \geq \epsilon \right\} < \epsilon. \quad (7)$$

Since $(a_n C_n^{\text{sym}}, b_n Z_n^{\text{sym}}) \xrightarrow{d} (C, Z)$ by assumption, after decreasing β if necessary, (7) implies that

$$\sup_n \mathbf{P} \left\{ \sup_{x,y: \text{Dist}_n^{\text{sym}}(x,y) \leq \beta/a_n} \{b_n |Z_n^{\text{sym}}(x) - Z_n^{\text{sym}}(y)|\} > \epsilon \right\} < \epsilon.$$

Since T_n^{sym} is isomorphic to a subspace of $\mathcal{T}_{C_n^{\text{sym}}}$, we have

$$\sup_{v,w \in T_n^{\text{sym}}: \text{dist}(v,w) \leq \beta/a_n} b_n |\ell_n^{\text{sym}}(v) - \ell_n^{\text{sym}}(w)| \leq \sup_{x,y: \text{Dist}_n^{\text{sym}}(x,y) \leq \beta/a_n} \{b_n |Z_n^{\text{sym}}(x) - Z_n^{\text{sym}}(y)|\},$$

and the result follows. \square

Proposition 11. *Under the hypotheses of Theorem 2, the family $(b_n Z_n)$ is tight.*

Proof. Fix $\epsilon > 0$, let $\beta < \epsilon$ be such that the bound of Lemma 10 holds and let $\alpha = \alpha(\beta)$ be such that the bound of Lemma 9 holds. Let $t = T_n$ and $N = 2|T_n| - 2$; we assume n is large enough that $\alpha N \geq 2$. Applying Lemma 8 with $k = \alpha N$, $K = \beta/a_n$ and $M = \epsilon/b_n$, we get

$$\begin{aligned} & \mathbf{P} \left\{ \sup_{|i-j| \leq \alpha N} |\ell_n(\theta_n(i)) - \ell_n(\theta_n(j))| > \epsilon/b_n \right\} \\ & \leq \mathbf{P} \left\{ \sup_{|i-j| \leq \alpha N} \text{dist}(\theta_n(i), \theta_n(j)) > \beta/a_n \right\} + \mathbf{P} \left\{ \sup_{\text{dist}(v,w) \leq \beta/a_n} |\ell_n^{\text{sym}}(v) - \ell_n^{\text{sym}}(w)| > \epsilon/b_n \right\}. \end{aligned}$$

By Lemmas 9 and 10, it follows that

$$\limsup_n \mathbf{P} \left\{ \sup_{|i-j| \leq \alpha N} |\ell_n(\theta_n(i)) - \ell_n(\theta_n(j))| > \epsilon/b_n \right\} \leq \beta + \epsilon \leq 2\epsilon. \quad (8)$$

For any $x, y \in [0, 1]$, by the triangle inequality, we have

$$|Z_n(x) - Z_n(y)| \leq |\ell_n(\lfloor xN \rfloor) - \ell_n(\lfloor yN \rfloor)| + |Z_n(x) - \ell_n(\lfloor xN \rfloor)| + |Z_n(y) - \ell_n(\lfloor yN \rfloor)|.$$

By definition of Z_n this yields

$$|Z_n(x) - Z_n(y)| \leq |\ell_n(\lfloor xN \rfloor) - \ell_n(\lfloor yN \rfloor)| + |\ell_n(\lceil xN \rceil) - \ell_n(\lfloor xN \rfloor)| + |\ell_n(\lceil yN \rceil) - \ell_n(\lfloor yN \rfloor)|.$$

Since $\alpha N \geq 2$, if $|x - y| \leq \alpha/2$ then $|\lfloor xN \rfloor - \lfloor yN \rfloor| \leq \alpha N$, so

$$\sup_{|x-y| \leq \alpha/2} |Z_n(x) - Z_n(y)| \leq 3 \sup_{|i-j| \leq \lfloor \alpha N \rfloor} |\ell_n(\theta_n(i)) - \ell_n(\theta_n(j))|.$$

Together with Equation (8), this establishes the requisite tightness. \square

Proof of Theorem 2. For $n \geq 1$ write \mathcal{L}_n for the law of $(a_n C_n, b_n Z_n)$, so \mathcal{L}_n is a Borel probability measure on $C([0, 1], \mathbb{R}^2)$. Proposition 11 and (6) together imply that the family $(\mathcal{L}_n, n \geq 1)$ is tight. To complete the proof, we establish convergence of finite-dimensional distributions by showing that for any $0 \leq t_1 < t_2 < \dots < t_m \leq 1$ and any bounded Lipschitz function $F : (\mathbb{R}^2)^m \rightarrow \mathbb{R}$,

$$\mathbf{E} \left[F \left((a_n C_n(t_i), b_n Z_n(t_i))_{1 \leq i \leq m} \right) \right] \rightarrow \mathbf{E} \left[F \left((C(t_i), Z(t_i))_{1 \leq i \leq m} \right) \right] \quad (9)$$

For the remainder of the proof, fix F and $(t_i, 1 \leq i \leq m)$ as above, let $\|F\|_\infty$ be the uniform norm of F , and let $\|F\|_{\text{Lip}}$ be the Lipschitz constant of F with respect to this norm.

By [6, Theorem 8.2], since $(\mathcal{L}_n, n \geq 1)$ is tight, for all $\delta > 0$ there is $\alpha = \alpha(\delta)$ such that

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left\{ \sup_{x, y \in [0, 1], |x-y| \leq \alpha} (a_n |C_n(x) - C_n(y)| + b_n |Z_n(x) - Z_n(y)|) > \delta \right\} < \delta. \quad (10)$$

We write $(A_n, n \geq 1)$ for the events whose probabilities are bounded in (10).

Since C and Z are almost surely uniformly continuous, by decreasing $\alpha(\delta)$ if necessary we may additionally ensure that

$$\mathbf{P} \left\{ \sup_{x, y \in [0, 1], |x-y| \leq \alpha} (|C(x) - C(y)| + |Z(x) - Z(y)|) > \delta \right\} < \delta. \quad (11)$$

Let $(U_i, i \geq 1)$ be independent Uniform $[0, 1]$ random variables. Fix $k \geq 1$ and let $(U_i^\uparrow, 1 \leq i \leq k)$ be the increasing reordering of U_1, \dots, U_k . Then fix $\delta > 0$, let $\alpha = \alpha(\delta)$ be as above, and let $j > 2/\alpha$ be large enough that

$$\mathbf{P} \left\{ \max_{1 \leq i \leq j} \left| U_i^\uparrow - \frac{i}{j} \right| \geq \frac{\alpha}{2} \right\} < \delta.$$

Now choose integers k_1, \dots, k_m so that for $1 \leq i \leq m$, $|k_i/j - t_i| < \alpha/2$; this is possible since $j > 2/\alpha$. It follows that

$$\mathbf{P} \left\{ \max_{1 \leq i \leq m} |U_{k_i}^\uparrow - t_i| \geq \alpha \right\} < \delta. \quad (12)$$

Let B and C be the events whose probabilities are bounded in (11) and (12). Writing $E_n = (A_n \cup B \cup C)^c$, we then have $\liminf_n \mathbf{P} \{E_n\} > 1 - 3\delta$. When E_n occurs,

$$\left| F \left((a_n C_{T_n}(t_i), b_n Z_{T_n}(t_i))_{1 \leq i \leq m} \right) - F \left((a_n C_{T_n}(U_{k_i}^\uparrow), b_n Z_{T_n}(U_{k_i}^\uparrow))_{1 \leq i \leq m} \right) \right| \leq \delta \|F\|_{\text{Lip}},$$

so for n sufficiently large,

$$\left| \mathbf{E} \left[F \left((a_n C_n(t_i), b_n Z_n(t_i))_{1 \leq i \leq m} \right) \right] - \mathbf{E} \left[F \left((a_n C_n(U_{k_i}^\uparrow), b_n Z_n(U_{k_i}^\uparrow))_{1 \leq i \leq m} \right) \right] \right| \leq \delta \|F\|_{\text{Lip}} + 6\delta \|F\|_{\infty}. \quad (13)$$

On E_n , we also have $|C(t_i) - C(U_{k_i}^\uparrow)| + |Z(t_i) - Z(U_{k_i}^\uparrow)| < \delta$, so it likewise follows that

$$\left| \mathbf{E} \left[F \left((C(t_i), Z(t_i))_{1 \leq i \leq m} \right) \right] - \mathbf{E} \left[F \left((C(U_{k_i}^\uparrow), Z(U_{k_i}^\uparrow))_{1 \leq i \leq m} \right) \right] \right| \leq \delta \|F\|_{\text{Lip}} + 6\delta \|F\|_{\infty}. \quad (14)$$

Finally, Lemma 7 implies that

$$\mathbf{E} \left[F \left((a_n C_n(U_{k_i}^\uparrow), b_n Z_n(U_{k_i}^\uparrow))_{1 \leq i \leq m} \right) \right] \rightarrow \mathbf{E} \left[F \left((C(U_{k_i}^\uparrow), Z(U_{k_i}^\uparrow))_{1 \leq i \leq m} \right) \right]$$

as $n \rightarrow \infty$. Together with (13) and (14), this yields (9) and completes the proof. \square

For use in the next section, we note two further straightforward points related to convergence of processes built from trees. For a tree t with $|V(t)| = n \geq 1$, and $q \in \mathbb{Q}$ a fixed type, define $\Lambda_t^{(q)}(i/n)$ as the number of times the contour process visits a vertex of type q before time i . More formally, for $0 \leq i \leq n$, set

$$\Lambda_t^{(q)}(i/n) = |\{0 \leq j < i : s(\theta(j)) = q\}|.$$

Then extend the domain of definition of $\Lambda_t^{(k)}$ to $[0, 1]$ by linear interpolation. The first proposition is an immediate consequence that if $T = (T, d)$ is a random labeled tree with valid law ν , then the law of the underlying plane tree T is symmetric; its proof is omitted.

Proposition 12. *For each $n \geq 1$ let $T_n = (T_n, D_n)$ be a random labeled multitype plane tree whose law ν_n is valid, and let T_n^{sym} have law ν_n^{sym} . Fix any type $q \in \mathbb{Q}$. If there exists a $C([0, 1], \mathbb{R})$ -valued random process $\Lambda^{(q)}$ such that*

$$|V(T_n^{\text{sym}})|^{-1} \Lambda_{T_n^{\text{sym}}}^{(q)} \xrightarrow{d} \Lambda^{(q)},$$

then also

$$|V(T_n)|^{-1} \Lambda_{T_n}^{(q)} \xrightarrow{d} \Lambda^{(q)}.$$

Next, fix a labeled plane tree $t = (t, d)$ and list the vertices of t in lexicographic order with respect to their Ulam-Harris labels as $\emptyset = v_0, v_1, \dots, v_{|t|-1}$, and write $v_{|t|} = v_0$ for convenience. The height process H_t of t is the $C([0, 1], \mathbb{R})$ function defined as follows. For integers $0 \leq i \leq |t|$ let $H_t(i/|t|) = \text{dist}_t(\emptyset, v_i)$; then extend to $[0, 1]$ by linear interpolation. Similarly, define $S_t : [0, 1] \rightarrow \mathbb{R}$ by taking $S_t(i/|t|) = \ell_t(\theta_t(i))$ for $0 \leq i \leq |t|$ and extending to $[0, 1]$ by linear interpolation.

Proposition 13. *Let $(T_n, n \geq 1)$ be random labelled plane trees. If*

$$(a_n H_{T_n}, b_n S_{T_n}) \xrightarrow{d} (C, Z),$$

for the uniform topology with $a_n \rightarrow 0$ and $a_n \cdot |T_n| \rightarrow \infty$, then

$$(a_n C_{T_n}, b_n Z_{T_n}) \xrightarrow{d} (C, Z)$$

also for the uniform topology.

Proof. We roughly follow the argument from Section 1.6 of [10], but must modify it slightly to handle the label process. Again fix a labeled plane tree $t = (t, d)$ and list the vertices of t in lexicographic order as $v_0, v_1, \dots, v_{|t|-1}$, writing $v_{|t|} = v_0$ for convenience. For $0 \leq i < |t|$ let $j(i) = j_t(i) = 2i - \text{dist}_t(\emptyset, v_i)$, and set $j(|t|) = 2|t| - 2$. Also, for $0 \leq s \leq 1$ let $\varphi_t(s) = |t|^{-1} \cdot \sup\{i : j_t(i) \leq s(2|t| - 2)\}$.

It is straightforward to verify the following facts.

- For all $0 \leq i < t$, $j(i) = \inf\{j \geq 0 : \theta_t(j) = v_i\}$.
- For all $0 \leq i < t$, for $1 \leq k < j(i+1) - j(i)$, the vertex $\theta_t(j(i) + k)$ is the parent of $\theta_t(j(i) + k - 1)$, and if $i < |t| - 1$ then $\theta_t(j(i+1))$ is the child of $\theta_t(j(i+1) - 1)$. In particular,

$$\text{dist}_t(\theta_t(j(i)), \theta_t(j(i) + k)) = k$$

$$\text{for } 1 \leq k < j(i+1) - j(i).$$

It follows that if $s \in (0, 1)$ and $j(i) \leq s(2|t| - 2) < j(i+1)$ then both $C_t(s)$ and $H_t(\varphi(s))$ lie in the interval

$$[\text{dist}_t(\emptyset, v_{i+1}) - 1, \text{dist}_t(\emptyset, v_i) + 1],$$

so

$$\sup_{0 \leq s \leq 1} |C_t(s) - H_t(\varphi(s))| \leq 2 + \max_{0 \leq i < |t|} \text{dist}_t(v_i, v_{i+1}). \quad (15)$$

Writing $\alpha_t = \max_{0 \leq i < |t|} \text{dist}_t(v_i, v_{i+1})$, it likewise follows that

$$\sup_{0 \leq s \leq 1} |Z_t(s) - S_t(\varphi(s))| \leq \max \left(|\ell_t(u) - \ell_t(v)| : \text{dist}_t(u, v) \leq \max_{0 \leq i < |t|} \text{dist}_t(v_i, v_{i+1}) \right). \quad (16)$$

Also, from the definition we straightforwardly have

$$(2|t| - 2) \cdot \sup_{0 \leq s \leq 1} |\varphi_t(s) - s| \leq 4 + \sup_{0 \leq i \leq |t|} |j(i) - 2i| = 4 + \text{height}(t). \quad (17)$$

We now turn to asymptotics. Assume for notational convenience that $|T_n| = n$. If $a_n H_{T_n} \xrightarrow{d} C$ then the fact that C is continuous implies that

$$a_n \cdot \max_{0 \leq i < n} \text{dist}_{T_n}(v_i, v_{i+1}) \rightarrow 0$$

in probability, and together with the fact that $a_n|\mathcal{T}_n| \rightarrow \infty$ implies that $\text{height}(\mathcal{T}_n)/n \rightarrow 0$ in probability. Using the first of these bounds in (15) gives that

$$a_n \sup_{0 \leq s \leq 1} |C_{\mathcal{T}_n}(s) - H_{\mathcal{T}_n}(\varphi(s))| \rightarrow 0$$

in probability; using the second in (17) gives that

$$\sup_{0 \leq s \leq 1} |\varphi_{\mathcal{T}_n}(s) - s| \rightarrow 0$$

in probability. Finally, again using the first bound in (16), and exploiting the convergence of $b_n S_{\mathcal{T}_n}$ to the continuous process Z as in the proof of Lemma 10, it follows that

$$b_n \sup_{0 \leq s \leq 1} |Z_{\mathcal{T}_n}(s) - S_{\mathcal{T}_n}(\varphi(s))| \rightarrow 0$$

in probability. The last three convergence results together imply that $(a_n H_{\mathcal{T}_n}, b_n S_{\mathcal{T}_n})$ and $(a_n C_{\mathcal{T}_n}, b_n Z_{\mathcal{T}_n})$ must have the same limit. \square

4. CONVERGENCE OF RANDOM NON-BIPARTITE BOLTSMANN PLANAR MAPS

4.1. Definitions around Theorem 1.

4.1.1. *Brownian tree, Brownian snake, Brownian map.* In the rest of this section, we denote $\mathbf{e} = (\mathbf{e}(s), 0 \leq s \leq 1)$ a standard Brownian excursion. Recall the construction given in Section 2.2. The random tree $(\mathcal{T}_{\mathbf{e}}, \text{Dist}_{\mathbf{e}})$ was introduced in [3] and is called the *Brownian Continuum Random Tree*.

Next, conditionally given \mathbf{e} , let $Z_{\mathbf{e}} = (Z_{\mathbf{e}}(s), 0 \leq s \leq 1)$ be a centred Gaussian process such that $Z(0) = 0$ and for $0 \leq s \leq t \leq 1$,

$$\text{Cov}(Z_{\mathbf{e}}(s), Z_{\mathbf{e}}(t)) = \inf\{\mathbf{e}(u), \text{ for } s \leq u \leq t\}.$$

We may and shall assume $Z_{\mathbf{e}}$ to be a.s. continuous; see [10, Section 3] for a more detailed description of the construction of the pair $(\mathbf{e}, Z_{\mathbf{e}})$, which is called the *Brownian snake*. It can be checked that almost surely, for all $x, y \in [0, 1]$, if $x \sim_{\mathbf{e}} y$ then $Z_{\mathbf{e}}(x) = Z_{\mathbf{e}}(y)$. Therefore, a.s the pair $(\mathbf{e}, Z_{\mathbf{e}})$ is a tree-like path.

To construct the Brownian map, we further need the following. For $x, y \in [0, 1]$, let

$$D^\circ(x, y) = D^\circ(y, x) = Z_{\mathbf{e}}(x) + Z_{\mathbf{e}}(y) - 2 \max\left(\min_{q \in [x, y]} Z_{\mathbf{e}}(q), \min_{q \in [y, 1] \cup [0, x]} Z_{\mathbf{e}}(q)\right). \quad (18)$$

Next let D^* be the largest pseudo-distance on $[0, 1]$ such that $D^* \leq D^\circ$, and such that for all $x, y \in [0, 1]$, if $x \sim_{\mathbf{e}} y$, then $D^*(x, y) = 0$. Then, let $M = [0, 1]/\{D^* = 0\}$ and let d^* be the push-forward of D^* to M . Finally, let λ be the push-forward of Lebesgue measure on $[0, 1]$ to M . The *Brownian map* is the (law of the) triple (M, d^*, λ) .

4.1.2. *Gromov-Hausdorff and Gromov-Hausdorff-Prokhorov distance.* We give in this section the definition of Gromov-Hausdorff and Gromov-Hausdorff-Prokhorov distances and refer the readers to [9] and [19, Section 6] for details.

Let X and X' be two compact metric spaces. The *Gromov-Hausdorff distance*¹ $d_{GH}(X, X')$ between X and X' is defined by

$$d_{GH}(X, X') = \inf_{\phi, \phi'} \delta_H(\phi(X), \phi(X')),$$

where the infimum is taken over all isometries $\phi : X \rightarrow Z$ and $\phi' : X' \rightarrow Z$ into a common metric space Z and where δ_H denotes the classical Hausdorff distance.

Let $\mathbf{X} = (X, \mu)$ and $\mathbf{X}' = (X', \mu')$ be two compact measured metric spaces (that is X and X' are two compact metric spaces and μ and μ' are Borel probability measures on X

¹We define here in fact a pseudo-distance and we should consider instead isometric classes of compact metric spaces to be perfectly rigorous.

and X' respectively). The *Gromov-Hausdorff-Prokhorov distance* $d_{GHP}(X, X')$ between X and X' is defined by

$$d_{GHP}(X, X') = \inf_{\phi, \phi'} \left(\delta_H(\phi(X), \phi(X')) \wedge \delta_P(\phi_*\mu, \phi_*\mu') \right),$$

where the infimum is taken over all isometries $\phi : X \rightarrow Z$ and $\phi' : X' \rightarrow Z$ into a common metric space Z and where δ_P denotes the Prokhorov distance between two probability measures and $\phi_*\mu$ and $\phi_*\mu'$ denote the push-forwards of μ and μ' by ϕ and ϕ' .

4.2. Planar maps and Boltzmann distribution. All maps considered in this section are *rooted*, meaning that an edge is marked and oriented. This edge is called the root edge and its tail is the *root vertex*. In addition to their rooting, maps can also be *pointed* meaning that an additional vertex is distinguished. A map M rooted at an oriented edge e and pointed at a vertex v^\bullet is denoted (M, e, v^\bullet) . The set of rooted maps and of rooted and pointed maps are respectively denoted \mathcal{M} and \mathcal{M}^\bullet . For $n \geq 1$, we denote by \mathcal{M}_n and \mathcal{M}_n^\bullet the subsets of \mathcal{M} and \mathcal{M}^\bullet consisting of maps with n vertices, respectively. We assume that \mathcal{M} and \mathcal{M}^\bullet both contain the “vertex map” \dagger , which consists of a single vertex, no edge, and a single face of degree 0.

Following [14] and [17], we introduce the *Boltzmann distribution* on \mathcal{M} defined as follows. Let $\mathbf{q} = (q_1, q_2, q_3, \dots)$ be a sequence of non-negative real numbers. For $m \in \mathcal{M}$ or $m \in \mathcal{M}^\bullet$, we define the weight of m by

$$W_{\mathbf{q}}(m) = \prod_{f \in f(m)} q_{\deg(f)},$$

where $f(m)$ is the set of faces of m ; by convention $W_{\mathbf{q}}(\dagger) = 1$.

For $n \geq 1$ let $\mathcal{Z}_{\mathbf{q}, n} = \sum_{m \in \mathcal{M}_n} W_{\mathbf{q}}(m)$ and let $\mathcal{Z}_{\mathbf{q}, n}^\bullet = \sum_{m \in \mathcal{M}_n^\bullet} W_{\mathbf{q}}(m) = n\mathcal{Z}_{\mathbf{q}, n}$. These quantities are both finite since \mathcal{M}_n^\bullet is finite. Then define probability measures $\mathbb{P}_{\mathbf{q}}$ and $\mathbb{P}_{\mathbf{q}}^\bullet$ on \mathcal{M}_n and \mathcal{M}_n^\bullet , respectively, by setting, for $m \in \mathcal{M}_n$ and $m^\bullet \in \mathcal{M}_n^\bullet$,

$$\mathbb{P}_{\mathbf{q}, n}(m) = \frac{W_{\mathbf{q}}(m)}{\mathcal{Z}_{\mathbf{q}, n}} \quad \text{and} \quad \mathbb{P}_{\mathbf{q}}^\bullet(m^\bullet) = \frac{W_{\mathbf{q}}(m^\bullet)}{\mathcal{Z}_{\mathbf{q}, n}^\bullet}.$$

$$\mathcal{Z}_{\mathbf{q}}^\bullet = \sum_{m^\bullet \in \mathcal{M}^\bullet} W_{\mathbf{q}}(m^\bullet) = \sum_{n \geq 1} \sum_{m^\bullet \in \mathcal{M}_n^\bullet} n W_{\mathbf{q}}(m^\bullet).$$

From now until the end of the paper, we assume the following holds.

Assumption 1. The sequence \mathbf{q} has finite support, and there exists an odd integer $p \geq 3$ such that $q_p > 0$.

Note that for any $c > 0$, with $c\mathbf{q} = (cq_1, cq_2, \dots)$, we have $\mathbb{P}_{\mathbf{q}, n} = \mathbb{P}_{c\mathbf{q}, n}$ and $\mathbb{P}_{\mathbf{q}, n}^\bullet = \mathbb{P}_{c\mathbf{q}, n}^\bullet$. As proved in the Appendix of [8], under Assumption 1, there is $c > 0$ such that $c\mathbf{q}$ is regular critical in the sense of [17, Definition 1]. Together these facts imply that, in studying $\mathbb{P}_{\mathbf{q}, n}$ and $\mathbb{P}_{\mathbf{q}, n}^\bullet$ we may assume \mathbf{q} is itself regular critical; we make this assumption henceforth.

The goal of this section is to prove the following result.

Theorem 14. *Let $(\mathbf{M}_n, n \geq 1)$ be random rooted maps with \mathbf{M}_n having law $\mathbb{P}_{\mathbf{q}, n}$. Denote by $\text{dist}_{\mathbf{M}_n}$ the graph distance on \mathbf{M}_n and by μ_n the uniform probability distribution on $V(\mathbf{M}_n)$. Then there exists a constant $C_{\mathbf{q}}$ such that, as n goes to infinity,*

$$\left(V(\mathbf{M}_n), \frac{C_{\mathbf{q}}}{n^{1/4}} d_{\mathbf{M}_n}, \mu_n \right) \xrightarrow{d} (M, d^*, \lambda),$$

for the Gromov-Hausdorff-Prokhorov topology, where (M, d^*, λ) is the Brownian map.

Remarks.

★ If there exists an odd integer $p \geq 3$ such that $q_p > 0$ and $q_j = 0$ for all $j \neq p$, then clearly \mathbf{q} satisfies Assumption 1. In this case $\mathbb{P}_{\mathbf{q},n}$ is the uniform distribution on the set of p -angulations with n vertices. Thus, Theorem 1 is a direct corollary of Theorem 14.

★ The pushforward of $\mathbb{P}_{\mathbf{q},n}^\bullet$ to \mathcal{M}_n obtained by forgetting the marked vertex is $\mathbb{P}_{\mathbf{q},n}$, so we may and will prove Theorem 14 for \mathbf{M}_n distributed according to $\mathbb{P}_{\mathbf{q},n}^\bullet$ rather than $\mathbb{P}_{\mathbf{q},n}$.

To prove this theorem, we rely on the method introduced by Le Gall in [12, Section 8]. This approach exploits distributional symmetries of the Brownian map (which can be deduced from the fact that the ensemble of quadrangulations has the Brownian map as their scaling limit [12, 20]) to simplify the task of proving convergence for other ensembles.

At a high level, to apply the method, three points need to be checked. First, \mathbf{M}_n must be encoded by a labeled tree such that vertices of the map are in correspondence with a subset of vertices of the tree and such that the labels on the vertices of the tree encode certain metric properties of the map. Second, the contour and label processes of the labeled trees encoding the sequence of maps should converge (once properly rescaled) to the Brownian snake, (defined in Section 4.1.1). Third, the vertex with minimum label in the tree must correspond to a vertex in the map whose distribution is asymptotically uniform on $V(\mathbf{M}_n)$ as n goes to infinity.

In our setting, the first point is achieved by the Bouttier-di Francesco-Guitter bijection, which we recall in the next section. The third one, addressed in Section 4.4, is a direct consequence of a result by Miermont. Proving that the second point holds (its precise statement will be given in Proposition 18) is the main new contribution of this section.

With these three ingredients, we conclude the proof of Theorem 14 in Section 4.5.

4.3. The Bouttier-di Francesco-Guitter bijection. In this section, we describe the Bouttier-di Francesco-Guitter bijection [7], roughly following the presentation given in [8].

Let (M, e, v^\bullet) be an element of \mathcal{M}^\bullet . Let e^- and e^+ be respectively the tail and the head of e . Three cases can occur: either $d_M(s, e^-) = d_M(s, e^+)$, $d_M(s, e^-) = d_M(s, e^+) + 1$ or $d_M(s, e^-) = d_M(s, e^+) - 1$. Depending on which case occurs, we say that M is respectively *null*, *negative* or *positive*. The set of null (resp. positive and negative) pointed and rooted maps is denoted \mathcal{M}^0 (resp. \mathcal{M}^+ and \mathcal{M}^-). By convention, we let $\dagger \in \mathcal{M}^+$. Reversing the orientation of the root edge gives a bijection between the sets $\mathcal{M}^+ \setminus \{\dagger\}$ and \mathcal{M}^- . Thus, in the following, we focus only on the sets \mathcal{M}^+ and \mathcal{M}^0 .

We now introduce the class of decorated trees or *mobiles* which appear in the bijection.

Definition 15. A mobile $t = (t, d)$ is a 4-type rooted plane labeled tree which satisfies the following constraints.

- (i) Vertices at even generations are of type 1 or 2 and vertices at odd generations are of type 3 or 4.
- (ii) Each child of a vertex of type 1 is of type 3.
- (iii) Each non-root vertex of type 2 has exactly one child of type 4 and no other child. If the root vertex is of type 2, it has exactly two children, both of type 4.

The labelling d is an admissible labeling of t , meaning that the following hold.

- (1) If the root is of type 1, vertices of type 1 and 3 are labeled by integers and vertices of type 2 and 4 by half-integers.
- (2) If the root is of type 2, vertices of type 2 and 4 are labeled by integers and vertices of type 1 and 3 by half-integers.
- (3) For all vertices u of type 3 or 4, for every $i = 0, \dots, k_t(u)$,

$$\begin{cases} \ell(u(i+1)) \geq \ell(ui) - 1 & \text{if } u(i+1) \text{ is of type 1,} \\ \ell(u(i+1)) \geq \ell(ui) & \text{if } u(i+1) \text{ is of type 2.} \end{cases}$$

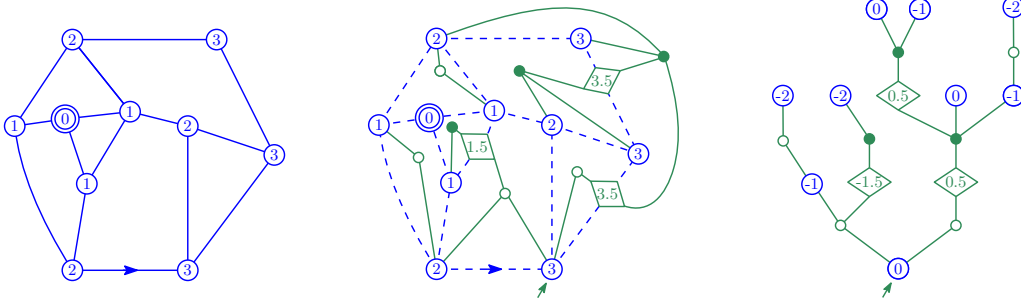


FIGURE 2. An example of the Bouttier-di Francesco-Guitter bijection. The original map (left), the construction of $\Phi(m)$ (middle) and the resulting mobile (right). Circle and square labeled vertices correspond respectively to vertices of type 1 and 2. Empty and filled unlabeled vertices correspond respectively to vertices of type 3 and 4.

where we use Ulam-Harris encoding together with the convention that $u(k_t(u) + 1)$ and $u0$ both denote the parent of u .

(4) For all vertices u of type 3 or 4, we have $\ell(u) = \ell(u0)$.

The set of mobiles such that the root is of type 1 (resp. of type 2) is denoted \mathbb{T}^+ (resp. \mathbb{T}^0). For $n \geq 1$, the respective subsets of \mathbb{T}^+ and \mathbb{T}^0 with $n - 1$ vertices of type 1 are denoted \mathbb{T}_n^+ and \mathbb{T}_n^0 . We also set $\mathbb{T}^- = \mathbb{T}^+$ and $\mathbb{T}_n^- = \mathbb{T}_n^+$. The notation is justified by Proposition 16, below.

We now give the construction which maps an element of \mathcal{M}^+ to an element of \mathbb{T}^+ and which is illustrated in Figure 2. (the construction for \mathcal{M}^0 being very similar, we refer the reader to the original paper [7] or to [8] for the details). First, label each vertex of M by $\ell_M(v) := d_M(v, v^\bullet)$. Then, for each edge of the map whose both extremities have the same label, say ℓ , add a “flag-vertex” in the middle of the edge and label it $\ell_M(f) := \ell + 1/2$. Call the resulting augmented map M' . Next, add a “face-vertex” f in each face of the map. Now, for each face of M' , considering its vertices in clockwise direction, each time a vertex v is immediately followed by a vertex w with smaller label (the introduction of flag-vertices ensure that any two adjacent vertices have different labels), draw an edge between v and the corresponding face-vertex. Erase all edges of M' .

The result of [7] ensures that the resulting map, denoted $\Phi(M)$, is in fact a spanning tree of the union of the set of face-vertices, of the set of flag-vertices and of the set of vertices $V(M) \setminus \{v^\bullet\}$. The tree $\Phi(M)$ inherits a planar embedding from M . To make it a rooted plane tree, we additionally root it at e^+ , and choose the first child of e^+ to be the face-vertex associated to the face on the left of (e^-, e^+) ; note that because M is positive, there always exists an edge in $\Phi(M)$ between e^+ and this face-vertex.

We assign types to the vertices of $\Phi(M)$ as follows. Vertices of M have type 1, and flag-vertices have type 2. Face-vertices have type 3 if their parent is of type 1 and have type 4 otherwise. This turns $\Phi(M)$ is a mobile, rooted at a vertex of type 1. For $v \in V(M) \setminus \{v^\bullet\}$ we denote the image of v in $\Phi(M)$ by $\phi(v)$.

Label the nodes of $\Phi(M)$ as follows. For u of type 1 or 2, let $\ell(u) = \ell_M(v) - \ell_M(e^+)$, this makes sense since v is a node of M' . Having rooted $\Phi(M)$, we give vertices of type 3 and 4 the same label as their parent. We now use these vertex labels to turn $\Phi(M)$ into a rooted labeled tree by giving each edge (u, ui) of $\Phi(M)$ the label $\ell(ui) - \ell(u)$.

The properties of this construction which are essential to our work appear in the following proposition. (Properties (i) and (ii) are contained in the above description and Property (iii) is Lemma 3.1 of [11]).

Proposition 16 (Properties of the Bouttier-di Francesco-Guitter bijection). *For each $n \geq 1$ and $\star \in \{-, 0, +\}$, Φ gives a bijection between \mathcal{M}_n^\star and \mathbb{T}_n^\star . For $(m, e, v^\bullet) \in \mathcal{M}^+$, write $\mathfrak{t} = (t, d)$ for the image of (m, e, v^\bullet) by Φ . Then*

- (i) *Elements of $V(m) \setminus \{v^\bullet\}$ are in bijection with vertices of type 1 in t .*
- (ii) *For all $v \in V(m) \setminus \{v^\bullet\}$, $\text{dist}_m(v, v^\bullet) = \ell_{\mathfrak{t}}(\Phi(v)) - \min_{x \in V(t)} \ell_{\mathfrak{t}}(x) + 1$.*
- (iii) *For all $u, v \in V(m) \setminus \{v^\bullet\}$,*

$$\text{dist}_m(u, v) \leq \ell_{\mathfrak{t}}(\Phi(v)) + \ell_{\mathfrak{t}}(\Phi(u)) - 2 \max \left(\check{\ell}_{\mathfrak{t}}(\Phi(u), \Phi(v)), \check{\ell}_{\mathfrak{t}}(\Phi(v), \Phi(u)) \right) + 2,$$

where $\check{\ell}_{\mathfrak{t}}$ is defined as follows. For $x, y \in \mathfrak{t}$, set $i_x = \inf\{i : \theta_{\mathfrak{t}}(i) = x\}$ and $i_y = \inf\{i : \theta_{\mathfrak{t}}(i) = y\}$. Then

$$\check{\ell}_{\mathfrak{t}}(x, y) = \begin{cases} \inf_{i \in [i_x, i_y]} \ell_{\mathfrak{t}}(\theta_{\mathfrak{t}}(i)) & \text{if } i_x \leq i_y \\ \inf_{i \in [i_x, 2|t|-2] \cup [0, i_y]} \ell_{\mathfrak{t}}(\theta_{\mathfrak{t}}(i)) & \text{if } i_y < i_x. \end{cases}$$

4.4. Convergence of labeled trees. This section is devoted to the study of random labeled trees obtained by applying the Bouttier-Di Francesco-Guitter bijection Φ to random maps distributed according to $\mathbb{P}_{\mathbf{q}, n}$.

Theorem 17. *There exist $a_{\mathbf{q}} > 0$ and $C_{\mathbf{q}} > 0$ such that the following holds. For $n \geq 1$ with $Z_{\mathbf{q}, n} > 0$ let M_n have law $\mathbb{P}_{\mathbf{q}, n}$, and let $\mathbb{T}_n = \Phi(M_n)$ be obtained by applying the Bouttier-di Francesco-Guitter bijection to M_n . Then as $n \rightarrow \infty$ along values with $Z_{\mathbf{q}, n} > 0$,*

$$\left(\frac{a_{\mathbf{q}}}{n^{1/2}} C_{\mathbb{T}_n}(s), \frac{C_{\mathbf{q}}}{n^{1/4}} Z_{\mathbb{T}_n}(s) \right)_{0 \leq s \leq 1} \xrightarrow{d} (\mathbf{e}(s), Z_{\mathbf{e}}(s))_{0 \leq s \leq 1},$$

for the topology of uniform convergence on $C([0, 1], \mathbb{R}^2)$. Moreover, for each $i \in \{1, 2, 3, 4\}$, there exists $\gamma_i \geq 0$ such that

$$\left(\frac{1}{n} \Lambda_{\mathbb{T}_n}^{(i)}(t), 0 \leq s \leq 1 \right) \xrightarrow{d} (\gamma_i s, 0 \leq s \leq 1),$$

for the topology of uniform convergence on $C([0, 1], \mathbb{R})$.

Theorem 17 is an immediate consequence of the following proposition. For $n \geq 1$ let $Z_{\mathbf{q}, n}^+ = \sum_{m \in \mathcal{M}_n^+} W_{\mathbf{q}}(m)$ and, if $Z_{\mathbf{q}, n}^+ > 0$ then define $\mathbb{P}_{\mathbf{q}, n}^+$ by

$$\mathbb{P}_{\mathbf{q}, n}^+(m) = \frac{W_{\mathbf{q}}(m)}{Z_{\mathbf{q}, n}^+}.$$

Likewise define $Z_{\mathbf{q}, n}^0, Z_{\mathbf{q}, n}^-$ and $\mathbb{P}_{\mathbf{q}, n}^0, \mathbb{P}_{\mathbf{q}, n}^-$.

Proposition 18. *There exist $a_{\mathbf{q}} > 0$ and $C_{\mathbf{q}} > 0$ such that for any symbol $\star \in \{-, 0, +\}$ the following holds. For $n \geq 1$ with $Z_{\mathbf{q}, n}^\star > 0$ let M_n^\star have law $\mathbb{P}_{\mathbf{q}, n}^\star$, and let $\mathbb{T}_n^\star = \Phi(M_n^\star)$ be obtained by applying the Bouttier-di Francesco-Guitter bijection to M_n^\star . Then as $n \rightarrow \infty$ along values with $Z_{\mathbf{q}, n}^\star > 0$,*

$$\left(\frac{a_{\mathbf{q}}}{n^{1/2}} C_{\mathbb{T}_n^\star}(s), \frac{C_{\mathbf{q}}}{n^{1/4}} Z_{\mathbb{T}_n^\star}(s) \right)_{0 \leq s \leq 1} \xrightarrow{d} (\mathbf{e}(t), Z_{\mathbf{e}}(s))_{0 \leq s \leq 1},$$

for the topology of uniform convergence on $C([0, 1], \mathbb{R}^2)$. Moreover, for each $i \in \{1, 2, 3, 4\}$, there exists $\gamma_i \geq 0$ such that

$$\left(\frac{1}{n} \Lambda_{\mathbb{T}_n^\star}^{(i)}(s), 0 \leq s \leq 1 \right) \xrightarrow{d} (\gamma_i s, 0 \leq s \leq 1),$$

for the topology of uniform convergence on $C([0, 1], \mathbb{R})$.

Proof. We provide details only for the case that $\star = +$, and briefly discuss the other cases at the end of the proof. Using the notation of Section 2.3, Proposition 4.5 of [8] gives the following description of the distribution of $\Phi(M_n^+) = \mathbb{T}_n^+ = (T_n^+, D_n^+)$.

(1) The tree T_n^+ is a 4-type Galton-Watson tree. Its root is of type 1, it has $n - 1$ vertices of type 1 and its offspring distribution $\zeta_{\mathbf{q}}^+$ is as follows.

- The support of $\zeta_{\mathbf{q}}^{(1)}$ is $\{0\} \times \{0\} \times \mathbb{Z}_+ \times \{0\}$, and for $k \geq 0$,

$$\zeta_{\mathbf{q}}^{(1)}(0, 0, k, 0) = \frac{1}{\mathcal{Z}_{\mathbf{q}}^+} \left(1 - \frac{1}{\mathcal{Z}_{\mathbf{q}}^+}\right)^k.$$

- $\zeta_{\mathbf{q}}^{(2)}(0, 0, 0, 1) = 1$.
- $\zeta_{\mathbf{q}}^{(3)}$ and $\zeta_{\mathbf{q}}^{(4)}$ are supported on $\mathbb{Z}_+ \times \mathbb{Z}_+ \times \{0\} \times \{0\}$, and for any $k, k' \geq 0$,

$$\zeta_{\mathbf{q}}^{(3)}(k, k', 0, 0) = \alpha_{\mathbf{q}} (\mathcal{Z}_{\mathbf{q}}^+)^k (\mathcal{Z}_{\mathbf{q}}^0)^{k'/2} \binom{2k + k' + 1}{k + 1} \binom{k + k'}{k} q_{2k+k'+2}$$

$$\zeta_{\mathbf{q}}^{(4)}(k, k', 0, 0) = \beta_{\mathbf{q}} (\mathcal{Z}_{\mathbf{q}}^+)^k (\mathcal{Z}_{\mathbf{q}}^0)^{k'/2} \binom{2k + k'}{k} \binom{k + k'}{k} q_{2k+k'+1},$$

where $\alpha_{\mathbf{q}}$ and $\beta_{\mathbf{q}}$ are the appropriate normalizing constants.

(2) Conditionally given T_n^+ , the labeling D_n^+ is uniformly distributed over all admissible labelings (see Definition 15).

The law $\nu_{\mathbf{q},n}^+$ of T_n^+ is valid (see section 2.3) but its displacements are not locally centered.

However, by Lemma 2 and Proposition 3 of [17], its symmetrization satisfies all the assumptions of Theorems 2 and 4 of [18]. The conclusion of those theorems is that there exist $a_{\mathbf{q}} > 0$ and $C_{\mathbf{q}} > 0$ such that, as $n \rightarrow \infty$ along values with $\mathcal{Z}_{\mathbf{q},n}^+ > 0$, with $T_n^{+, \text{sym}}$ distributed as $\nu_{\mathbf{q},n}^{+, \text{sym}}$, then

$$\left(\frac{a_{\mathbf{q}}}{n^{1/2}} H_{T_n^{+, \text{sym}}}(s), \frac{C_{\mathbf{q}}}{n^{1/4}} S_{T_n^{+, \text{sym}}}(s) \right)_{0 \leq s \leq 1} \xrightarrow{\text{d}} (\mathbf{e}(s), Z(s))_{0 \leq s \leq 1},$$

for the topology of uniform convergence on $\mathcal{C}([0, 1]^2)$, and for all $i \in \{1, 2, 3, 4\}$,

$$\left(\frac{1}{n} \Lambda_{T_n^{+, \text{sym}}}^{(i)}(s), 0 \leq s \leq 1 \right) \xrightarrow{\text{(p)}} (\gamma_i s, 0 \leq s \leq 1)$$

for the topology of uniform convergence on $\mathcal{C}([0, 1])$, for suitable constants $(\gamma_i, 1 \leq i \leq 4)$. By Proposition 13 and Theorem 2, the first convergence also holds with $H_{T_n^{+, \text{sym}}}$ and $S_{T_n^{+, \text{sym}}}$ replaced by $C_{T_n^+}$ and $Z_{T_n^+}$, respectively; by Proposition 12, the second convergence also holds with $T_n^{+, \text{sym}}$ replaced by T_n^+ . This completes the proof in the case that $\star = +$.

Since it suffices to flip the orientation of the root edge of a map sampled from $\mathbb{P}_{\mathbf{q},n}^+$ to simulate $\mathbb{P}_{\mathbf{q},n}^-$, the case that $\star = -$ reduces to the case that $\star = +$. In the case $\star = 0$, the pushforward $\nu_{\mathbf{q},n}^0$ of $\mathbb{P}_{\mathbf{q},n}^0$ by Φ is very similar to $\nu_{\mathbf{n},\mathbf{q}}^+$, the only exception being that the root has type 2 and has two children; a reprise of the above arguments (again using the results of results of [18] together with Theorem 2) again yields the result in this case. \square

4.5. Convergence of Boltzmann maps. We conclude the proof of Theorem 14 in this section. Our argument closely mimics that given in [12, Section 8] for the convergence of triangulations, so we only give the main steps of the proof. Let \mathbf{M}_n have law $\mathbb{P}_{\mathbf{q},n}^+$. We shall prove that

$$\left(V(\mathbf{M}_n), \frac{C_{\mathbf{q}}}{n^{1/4}} \text{dist}_{\mathbf{M}_n}, \mu_n \right) \xrightarrow{\text{d}} (M, d^*, \lambda)$$

in distribution for the Gromov-Hausdorff-Prokhorov topology, as $n \rightarrow \infty$. The same holds with \mathbf{M}_n having law $\mathbb{P}_{\mathbf{q},n}^0$ or $\mathbb{P}_{\mathbf{q},n}^-$, with essentially the same proof (with some minor modifications, as at the end of Section 4.4; we omit the details for these cases).

Fix (m, e, v) in \mathcal{M}^+ and write $\Phi(m, e, v) = \mathbf{t} = (t, d) \in \mathcal{T}^+$. For $i \leq j \in \{0, \dots, 2|t| - 2\}$ such that $\theta(i)$ and $\theta(j)$ are both of type 1, we define

$$\delta_{\mathbf{t}}(i, j) = \text{dist}_m(\Phi^{-1}(\theta(i)), \Phi^{-1}(\theta(j)))$$

and, writing $\ell = \ell_t$ for the vertex labeling of t ,

$$\delta_t^\circ(i, j) = \delta_t^\circ(j, i) = \ell(\theta(i)) + \ell(\theta(j)) - 2 \max \left(\min_{k \in \{i, \dots, j\}} \ell(\theta(k)), \min_{k \in \{j, \dots, 2|t| - 2\} \cup \{0, \dots, i\}} \ell(\theta(k)) \right) + 2.$$

It follows directly from (iii) of Proposition 16 that

$$\delta_t(i, j) \leq \delta_t^\circ(i, j). \quad (19)$$

Finally, we extend both δ_t and δ_t° to $[0, 2|t| - 2] \times [0, 2|t| - 2]$ by linear interpolation. The inequality in (19) readily extends to this whole set.

We write $\mathbf{T}_n = \Phi(\mathbf{M}_n)$ and $m_n = 2|\mathbf{T}_n| - 2$, and for $0 \leq r, s \leq 1$ we define

$$D_n(r, s) := \frac{C_{\mathbf{q}}}{n^{1/4}} \delta_{\mathbf{T}_n}(m_n r, m_n s) \quad \text{and} \quad D_n^\circ(r, s) := \frac{C_{\mathbf{q}}}{n^{1/4}} \delta_{\mathbf{T}_n}^\circ(m_n r, m_n s).$$

Recall the definition of D° given in (18). Since D° depends continuously on Z , Proposition 18 implies that $D_n^\circ \xrightarrow{d} D^\circ$; moreover this convergence holds jointly with that stated in Proposition 18. Together with the bound (19) this implies (see [11, Proposition 3.2]) that the family of laws of $(D_n, n \geq 1)$ is tight in the space of probability measures on $C([0, 1], \mathbb{R}^2)$. Hence, from any increasing sequence of positive integers, we can extract an increasing subsequence $(n_j)_{j \geq 1}$, such that, jointly with the convergence in Proposition 18, we have

$$\left(D_{n_j}^\circ(s, t), D_{n_j}(s, t) \right)_{0 \leq s \leq t \leq 1} \xrightarrow{d} (D^\circ(s, t), \tilde{D}(s, t))_{0 \leq s \leq t \leq 1}, \quad (20)$$

for some random process \tilde{D} taking values in $C([0, 1], \mathbb{R})$. We will show that this convergence holds without extracting a subsequence with $\tilde{D} = D^*$; from this the theorem follows just as in [12], since d^* is the push-forward of D^* to $[0, 1] \setminus \{D^* = 0\}$. By the Skorohod representation theorem, we may and will assume that the convergence along $(n_j)_{j \geq 1}$ in (20) and in Proposition 18 jointly hold almost surely. To prove that the distributional convergence holds without extracting a subsequence, it then suffices to prove that $\tilde{D} \stackrel{\text{a.s.}}{=} D^*$, where D^* is defined in Section 4.1.1; recall that D^* is a measurable function of D° .

By (19), necessarily $\tilde{D} \leq D^\circ$ almost surely. Since \tilde{D} satisfies the triangle inequality, it then follows from the definition of D^* that $\tilde{D} \leq D^*$ almost surely. By continuity, to show that $\tilde{D} \stackrel{\text{a.s.}}{=} D^*$ it is then enough to prove that for any U, V two independent uniform random variables in $[0, 1]$, independent of all the other random objects, we have $\tilde{D}(U, V) \stackrel{d}{=} D^*(U, V)$.

So let U, V be two such uniform random variables. List the vertices of type 1 in \mathbf{T}_n in lexicographic order as v_1, \dots, v_{n-1} . For $k \in \{1, \dots, n-1\}$, let $i_{\mathbf{T}_n}(k) = \inf\{i \geq 0, \theta_{\mathbf{T}_n}(i) = v_k\}$. Next, define $U_n = \lceil U \cdot (n-1) \rceil$ and $V_n = \lceil V \cdot (n-1) \rceil$ which are both uniformly distributed on $\{1, \dots, n-1\}$. It follows from the last statement of Proposition 18 that

$$\frac{i_{\mathbf{T}_n}(U_n)}{m_n} \xrightarrow{(p)} U \quad \frac{i_{\mathbf{T}_n}(V_n)}{m_n} \xrightarrow{(p)} V.$$

Hence,

$$\left| D_n(U, V) - \frac{C_{\mathbf{q}}}{n^{1/4}} \text{dist}_{M_n}(\phi^{-1}(v_{U_n}), \phi^{-1}(v_{V_n})) \right| \xrightarrow{(p)} 0. \quad (21)$$

Let now X_n, Y_n be two independent uniform vertices of \mathbf{M}_n . Then, the convergence stated in (20) and in (21) imply that $\frac{C_{\mathbf{q}}}{n_j^{1/4}} \text{dist}_{M_{n_j}}(X_{n_j}, Y_{n_j})$ also converges in distribution². Since,

²Observe that $\phi^{-1}(v_{U_n})$ and $\phi^{-1}(v_{V_n})$ are two uniform vertices of $\mathbf{M}_n \setminus \{v^*\}$ rather than two uniform vertices of \mathbf{M}_n . But, since $|V(\mathbf{M}_n)| = n$, the difference is asymptotically negligible.

$\text{dist}_{\mathbf{M}_n}(X_n, Y_n) \stackrel{d}{=} \text{dist}_{\mathbf{M}_n}(X_n, v^\bullet(\mathbf{M}_n))$, for all n , it follows that the limiting distribution of $\frac{C_{\mathbf{q}}}{n_j^{1/4}} \text{dist}_{\mathbf{M}_{n_j}}(X_{n_j}, Y_{n_j})$ is the same as the limiting distribution of

$$\frac{C_{\mathbf{q}}}{n^{1/4}} \text{dist}_{\mathbf{M}_n}(X_n, v^\bullet(\mathbf{M}_n)) = \frac{C_{\mathbf{q}}}{n^{1/4}} (Z_{T_n}(U_n) - \min Z_{T_n} + 1),$$

where the equality is a consequence of (ii) in Proposition 16 and of the definition of Z_{T_n} (see Section 2.1). This last quantity converges to $Z_e(U) - \inf(Z_e(s), 0 \leq s \leq 1)$. By [12, Corollary 7.3], $Z_e(U) - \inf(Z_e(s), 0 \leq s \leq 1) \stackrel{d}{=} D^*(U, V)$, and therefore $\tilde{D} \stackrel{\text{a.s.}}{=} D^*$ as required. \square

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LIST OF NOTATION AND TERMINOLOGY

$[t]$	Set of plane trees isomorphic to t as multitype rooted trees.	2
$[t]$	Set of plane trees isomorphic to t as multitype labeled rooted trees.	2
$\llbracket v, w \rrbracket$	The unique path between vertices v and w belonging to some plane tree.	10
$\text{branch}(t, u)$	Vertices of t which correspond to branchpoints of $t(u)$	6
C_t	The contour process of t ; $C_t : [0, 1] \rightarrow \mathbb{R}$	4
$\text{ctype}(v)$	The child type vector of v	3
$d(e)$	The displacement along edge e	2
$\text{Dist}_C(x, y)$	$\text{Dist}_C(x, y) = C_t(x) + C_t(y) - 2 \inf\{C_t(u), x \leq u \leq y\}$	4
$\text{dist}_t(v, w)$	Graph distance between v and w in t	3
$d(v)$	Pushforward of d to $t(v)$	6
$d(v)$	Modification of $d(v)$ so child displacements at branchpoints are zero.	6
H_t	The height process of tree t	14
$h(v)$	The distance $\text{dist}(\emptyset, v)$ from the root to v	3
$k_t(v)$	The number of children of v in t	2
Λ_t^q	Counts nodes of type q along the contour process.	13
$\ell(v)$	Sum of displacements on root-to- v path.	3
μ^{sym}	The symmetrization of measure μ	3
ν^{sym}	The symmetrization of measure ν	3
P_t	Set of vectors $(\sigma_v, v \in V(t))$, each $\sigma_v \in \mathfrak{S}_{k_t(v)}$	2
$P_{(t,u)}$	Permutations $\sigma \in P_t$ leaving ordering unchanged at $v \in \text{branch}(t, u)$	6
$s(v)$	The type of vertex v ; $s(v) \in \mathbb{Q}$	2
$\sigma(t)$	Tree $t' \in [t]$ obtained by reordering children of each node v using σ_v	2
$\sigma(t)$	Tree $t' \in [t]$ obtained by reordering children, child edge labels of v using σ_v	2
$\sigma(v)$	Image of v in $\sigma(t)$	2
Symmetric	Measure μ (resp. ν) on plane trees (resp. labeled plane trees) is symmetric if $\mu(t) = \mu(t')$ whenever $t' \in [t]$ (resp. if $\nu(t) = \nu(t')$ whenever $t' \in [t]$).	2
θ_t	The contour exploration of tree t ; takes values in $V(t)$	3
Tree-like path	A pair (ζ, f) with ζ an excursion, $f(0) = f(1) = 1$, and $\text{Dist}_\zeta(x, y) = 0 \Rightarrow f(x) = f(y)$	4
$t(v)$	The subtree of t spanned by vertices of v and their ancestors.	6
$U(v, v_i)$	Ulam-Harris label of node of $t(v)$ corresponding to v_i	6
Valid law	A law ν on labeled plane trees satisfying certain independence and symmetry properties.	3
Z_t	The label process of t ; $Z_t : [0, 1] \rightarrow \mathbb{R}$	4

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DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, 805 SHERBROOKE STREET WEST, MONTRÉAL, QUÉBEC, H3A 2K6, CANADA

LIX UMR 7161, ÉCOLE POLYTECHNIQUE, 1 RUE HONORÉ D'ESTIENNE D'ORVES, 91120 PALAISEAU, FRANCE.

E-mail address: louigi@math.mcgill.ca

E-mail address: albenque@lix.polytechnique.fr

URL: <http://www.math.mcgill.ca/~louigi/>

URL: <http://www.lix.polytechnique.fr/~albenque/>