Bisimplicial vertices in even-hole-free graphs

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Abstract

A *hole* in a graph is an induced subgraph which is a cycle of length at least four. A hole is called *even* if it has an even number of vertices. An *even-hole-free* graph is a graph with no even holes. A vertex of a graph is *bisimplicial* if the set of its neighbours is the union of two cliques. In this paper we prove that every even-hole-free graph has a bisimplicial vertex, which was originally conjectured by Reed.

1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. The complement, $\overline{G}$, of G is the graph with vertex set $V(G)$ and such that two vertices $u, v \in V(G)$ are adjacent in G if and only if they are non-adjacent in $\overline{G}$. A *clique* in G is a set of vertices, all pairwise adjacent. Let S be a subset of $V(G)$. We denote by $G|S$ the subgraph of G induced on S, and by $G \setminus S$ the subgraph of G induced on $V(G) \setminus S$. We say that S is *connected* if $G|S$ is connected. A *component* of S is a maximal subset S' of S such that $G|S'$ is connected. An *anticomponent* of S is a maximal subset S' of S such that $\overline{G}|S'$ is connected. The *neighbourhood* of S, denoted by $N_G(S)$ (or $N(S)$ when there is no risk of confusion), is S together with the set of all vertices of $V(G) \setminus S$ with a neighbour in S. If $S = \{v\}$, we write $N_G(v)$ instead of $N_G(\{v\})$ (and, respectively, $N(v)$ instead of $N(\{v\})$). For an induced subgraph H of G, we define $N(H)$ to be $N(V(H))$. The *non-neighbourhood* of S is the set $V(G) \setminus N(S)$. A vertex is called *bisimplicial (in G)* if its neighbourhood is the union of two cliques. Two disjoint subsets A, B of $V(G)$ are *complete* to each other if every vertex of A is adjacent to every vertex of B, and *anticomplete* to each other if no vertex of A is adjacent to any vertex of B. If $A = \{a\}$, we write "a is complete (anticomplete) to B" instead of "$\{a\}$ is complete (anticomplete) to B".

A *hole* in a graph is an induced subgraph which is a cycle of length at least four. An *antihole* in a graph is a hole in its complement. A hole is *even* if it has an even number of vertices (and, equivalently, edges), and *odd* otherwise. A graph is *even-hole-free* if it contains no even hole. Even-hole-free graphs were studied in [2] and are known to be recognizable in polynomial time ([1], [3]). However, the following conjecture of Reed has remained open [4]:

1.1 *Every non-null even-hole-free graph has a bisimplicial vertex.*

A graph G is called *odd-signable* if there exists a function $f : E(G) \to \{0, 1\}$ such that $\sum_{e \in E(H)} f(e)$ is odd for every hole H of G. It is natural to ask whether 1.1 is true if we replace "even-hole-free" by "odd-signable". The answer to this question is "no", and the six vertex graph which is the 1-skeleton of the cube is a counterexample.

The goal of this paper is to prove 1.1. However, for inductive arguments, it turns out to be helpful to consider a slightly stronger statement. Instead of just finding one bisimplicial vertex, we prove that every subgraph with certain properties contains one.

A set S of vertices in a graph G is called *dominating (in G)* if $N(S) = V(G)$, and *non-dominating* otherwise. An induced subgraph H of G is *dominating* if $V(H)$ is dominating, and *non-dominating* otherwise. We can now state our main theorem.

1.2 *Let G be an even-hole-free graph. Then both the following statements hold:*

1. *If H is a non-dominating hole in G, then some vertex of $V(G) \setminus N(H)$ is bisimplicial in G.*
2. *If K is a non-dominating clique in G of size at most two, then some vertex of $V(G) \setminus N(K)$ is bisimplicial in G.*

Clearly the second statement of 1.2 with $K = \emptyset$ implies 1.1. We remark that the second statement of 1.2 is false if we replace "at most two" by "at most three". The graph obtained from K_4 by choosing a vertex and subdividing once the edges incident with it is a counterexample.

Let us now outline the proof of 1.2. The proof uses induction. Let G be a graph such that 1.2 holds for all smaller graphs. First we suppose that G fails to satisfy the first statement, that is there is a non-dominating hole H in G, but there is no bisimplicial vertex in the non-neighbourhood of $V(H)$. Now the idea is to examine the neighbourhood of $V(H)$ and try to find what we call a "useful cutset" in G, that is, a subset C of $V(G)$ and an edge e with both ends in C such that

- $V(G) \setminus C$ is the disjoint union of two non-empty sets, L and R, anticomplete to each other
- $C \subseteq N(e)$ and the non-neighbourhood of e in the graph $G|(C \cup R)$ is a non-empty subset of the non-neighbourhood of $V(H)$ in G.

If we find such a cutset C, then it follows, from the minimality of G, that R contains a vertex v which is bisimplicial in $G|(C \cup R)$; and since L is anticomplete to R, it follows that v is a bisimplicial vertex of G, which is a contradiction.

Unfortunately, we do not always succeed in finding a useful cutset; sometimes we have to make do with a set C and a list $u_1, .., u_k, v_1, .., v_k$ of vertices of C (possibly with repetitions) where u_i is non-adjacent to v_i in G for every $1 \leq i \leq k$, such that:

- $V(G) \setminus C$ is the disjoint union of two non-empty sets, L and R, anticomplete to each other
- the graph G' obtained from $G|(R \cup C)$ by adding the edge $u_i v_i$ for every $1 \leq i \leq k$ is even-hole-free
- For some edge e of G', $C \subseteq N(e)$, and the non-neighbourhood of e in the graph G' is a non-empty subset of the non-neighbourhood of $V(H)$ in G
- if v is a bisimplicial vertex of G' contained in the non-neighbourhood of e, then v is bisimplicial in G.

Having found such a set C etc, the same argument as in the case of a "genuine" useful cutset leads to a contradiction.

So G satisfies the first statement of 1.2. Suppose it fails to satisfy the second. This means that there is a non-dominating clique K of size at most two in G with no bisimplicial vertex in its non-neighbourhood. An easy argument shows that there is a hole H of G such that K is included in $V(H)$. Since the first assertion of the theorem holds for G, we deduce that H is dominating in G. Now we can examine the structure of G relative to H, and again find variations on the idea of a useful cutset, such as the one described above, that lead to a contradiction. So G satisfies the second statement of 1.2 too. This completes the inductive proof.

For a graph G, we denote by $\chi(G)$ the chromatic number of G, and by $\omega(G)$, the size of the largest clique of G. Finally, we would like to point out the following easy corollary of 1.2:

1.3 *Let G be an even-hole-free graph. Then $\chi(G) \leq 2\omega(G) - 1$.*

Proof. By 1.2, some vertex v of G is bisimplicial, and therefore v has degree at most $2\omega - 2$. Now the result follows by deleting v and applying induction. ∎

2 Preliminaries

Let us start with some definitions. We say that P is a *path* in G if P is an induced connected subgraph of G, such that either P is a one-vertex graph, or two vertices of P have degree one, and all the others have degree two. (This definition is non-standard, but very convenient.) The *length* of a path is the number of edges in it. A path is called *even* if its length is even, and *odd* otherwise. Let the vertices of P be $p_1, \dots, p_k$ in order. Then p_1, p_k are called the *ends* of P (sometimes we say P is *from* p_1 *to* p_k or *between* p_1 *and* p_k), and the set $V(P) \setminus \{p_1, p_k\}$ is the *interior* of P and is denoted by P^*. For $1 \leq i < j \leq k$ we will write p_i-P-p_j or p_j-P-p_i to mean the subpath of P between p_i and p_j. Similarly, if H is a hole, and a, b and c are three vertices of H such that a is adjacent to b, then a-b-H-c is a path, consisting of a, and the subpath of $H \setminus \{a\}$ between b and c.

A *theta* in a graph G means an induced subgraph T of G with two nonadjacent vertices s, t and three paths P, Q, R, each between s, t, such that P, Q, R are disjoint apart from their ends, the union of every pair of them is a hole, and $T = P \cup Q \cup R$. A *prism* in G is an induced subgraph P in which there are three paths R_1, R_2, R_3, with the following properties:

- for $i = 1, 2, 3$, R_i has length > 0; let its ends be a_i, b_i
- R_1, R_2, R_3 are pairwise disjoint, and $V(P) = V(R_1 \cup R_2 \cup R_3)$
- for $1 \leq i < j \leq 3$, there are precisely two edges between $V(R_i)$ and $V(R_j)$, namely a_ia_j and b_ib_j.

An *even wheel* in G is an induced subgraph consisting of a hole H and a vertex $v \notin V(H)$ with an even number, and at least four, neighbours in $V(H)$.

It is easy to see that every theta, every prism, and every even wheel contains at least one even hole, and therefore

2.1 *No even-hole-free graph contains a theta, a prism or an even wheel.*

Let H be a hole in G and let $v \in V(G) \setminus V(H)$. We say that (with respect to H) v is

- a *leaf* if it has exactly one neighbour in $V(H)$,
- a *hat* if it has exactly two neighbours in $V(H)$ and they are adjacent,
- a *clone* if its neighbours in $V(H)$ form a two-edge subpath of H,
- a *pyramid* if v has exactly three neighbours in $V(H)$ and exactly one pair of them is an edge of H, and
- a *major vertex* if either three neighbours of v in $V(H)$ are pairwise non-adjacent, or $|V(H)| = 5$ and v is complete to $V(H)$.

If v is a leaf with respect to H and the neighbour of v in $V(H)$ is n_1, we say that v is a leaf *at* n_1. If v is a hat with neighbours n_1, n_2, then v is a hat *at* n_1n_2. If v is a clone with respect to H and the neighbours of v in $V(H)$ are n_1, n_2, n_3 where n_1 is non-adjacent to n_3, we say that v is a clone *at* n_2. Finally, if v is a pyramid with respect to H with neighbours n_1, n_2, n_3 in $V(H)$ where n_1 is adjacent to n_2, we say that v is a pyramid with *base* n_1n_2 and *apex* n_3.

2.2 *Let G be an even-hole-free graph and let H be a hole of G. Let v be a vertex of $V(G) \setminus V(H)$ with a neighbour in $V(H)$. Then v is either a leaf, or a hat, or a clone, or a pyramid, or a major vertex with respect to H.*

Proof. Let N be the set of neighbours of v in $V(H)$. We may assume that $|N| > 1$, no three vertices of N are pairwise non-adjacent, and if $|V(H)| = 5$, then v is not complete to $V(H)$, for otherwise the theorem holds. It follows that $|N| \leq 4$, and therefore by 2.1 $|N| \leq 3$.

Suppose $|N| = 2$ and write $N = \{n_1, n_2\}$. We may assume that n_1 is non-adjacent to n_2, for otherwise the theorem holds. But now the subgraph induced by G on $V(H) \cup \{v\}$ is a theta, contrary to 2.1.

Next assume that $|N| = 3$ and write $N = \{n_1, n_2, n_3\}$. Since no three vertices of N are pairwise non-adjacent, we may assume that n_1 is adjacent to n_2. If n_3 is anticomplete to $\{n_1, n_2\}$, then v is a pyramid with respect to H, so we may assume that n_3 is adjacent to n_2, say. Since H is a hole, n_3 is non-adjacent to n_1, and therefore n_1-n_2-n_3 is a two-edge subpath of H and v is a clone with respect to H. So if $|N| = 3$, the theorem holds. This completes the proof of 2.2. ∎

The following is a lemma that we use a number of times in the course of the proof.

2.3 *Let G be even-hole-free, let K be a clique in G, and let S be a subset of $V(G) \setminus K$. Assume that $V(G) \setminus (K \cup S)$ is the disjoint union of two sets, L and R, such that L is connected and anticomplete to R. Assume also that every vertex of K has a neighbour in L, and there is a vertex $a \in L$, such that S is complete to a and anticomplete to $L \setminus \{a\}$.*

Define the graph G' as follows. Let $V(G') = R \cup S \cup K$, and let $u, v \in V(G)$ be adjacent if and only if there is an odd path of G between them with interior in L.

Then G' is even-hole-free.

We remark that every two vertices of G' that are adjacent in G, are still adjacent in G'. Since S is anticomplete to $L \setminus \{a\}$ and K is a clique, it follows that every edge in $E(G') \setminus E(G)$ has one end in S and the other in K.

Proof. We observe that, since L is connected and every vertex of $K \cup S$ has a neighbour in L, it follows that for every $k' \in K$ and $s' \in S$, there is a path from k' to s' in G with interior in L. Assume for a contradiction that there is an even hole H in G'. Since $G|(K \cup R \cup S)$ is even-hole-free, it follows that at least one edge of H belongs to $E(G') \setminus E(G)$. So there exist two vertices k and s of H such that $k \in K$, $s \in S$, and k is adjacent to s in G' but not in G.

(1) If $k \in K$ is non-adjacent in G' to $s \in S$, then every path from k to s with interior in R is odd.

Let P be a path from k to s with interior in R and let Q be a path from k to s with interior in L. Since s, k are non-adjacent in G', it follows that Q is even. But now, since k-Q-s-P-k is not an even hole in G, it follows that P is odd. This proves (1).

(2) Let $k \in K$ and $s \in S$ be adjacent in G' and non-adjacent in G. Then every path from k to s in G with interior in R is even.

Let P be a path from k to s in G with interior in R and let Q be a path from k to s in G with interior in L. Since s, k are adjacent in G', it follows that Q is odd. But now, since k-Q-s-P-k is not an even hole in G, it follows that P is even. This proves (2).

(3) $|V(H) \cap (S \cup K)| > 2$.

If $V(H) \cap (K \cup S) = \{k, s\}$, then the graph induced by G on $V(H)$ is an odd path from k to s with interior in R, contrary to (2). This proves (3).

(4) Every vertex of K, incident with an edge of $E(G') \setminus E(G)$, is complete to S in G'.

Let $k_1s_1 \in E(G') \setminus E(G)$ for some $k_1 \in K$ and $s_1 \in S$, and let s_2 be in S. Since $k_1s_1 \in E(G') \setminus E(G)$, we deduce from the definition of G' that in G there exists an odd path P from k_1 to s_1 with interior in L. Since S is complete to a and anticomplete to $L \setminus a$, it follows that the neighbour of s_1 in P is a, and k_1-P-a-s_2 is an odd path from k_1 to s_2 with interior in L. But now, $k_1s_2 \in E(G')$, again by the definition of G'. This proves (4).

By (4) k is complete to S, and therefore $|V(H) \cap S| \leq 2$. Assume first that $|V(H) \cap S| = 2$, and let s' be the vertex of $V(H) \cap S$ different from s. Since H is a hole, s is non-adjacent to s' and $V(H) \setminus \{k, s, s'\}$ is included in R. Let P be the path $H \setminus \{k\}$. Now, since H is an even hole, P is even, and s-P-s'-a-s is an even hole in G, a contradiction. This proves that $V(H) \cap S = \{s\}$, and therefore, by (3) and since K is a clique, $|V(H) \cap K| = 2$. Let k' be the vertex of $V(H) \cap K$ different from k. Then k' is non-adjacent to s and $V(H) \setminus \{k, k', s\}$ is a subset of R. But then, since H is an even hole, the path $H \setminus \{k\}$ is even, contrary to (1). This completes the proof of 2.3. ▮

Finally, we show the following:

2.4 *Let G be a counterexample to 1.2 with $|V(G)|$ minimum, and assume that there exists a non-dominating clique K' of size at most two in G such that no vertex of $V(G) \setminus N(K')$ is bisimplicial in G. Then there exists a non-dominating clique K of size exactly two in G such that no vertex of $V(G) \setminus N(K)$ is bisimplicial in G.*

Proof. First we show that we may assume $|K'| = 1$. For suppose that $K' = \emptyset$. If G is a complete graph, then every vertex of G is bisimplicial, contrary to the assumption, so there is a non-dominating vertex k''. Now $K'' = \{k''\}$ is a non-dominating clique of size one in G such that no vertex of $V(G) \setminus N(K'')$ is bisimplicial in G. We therefore assume that $K' = \{k'\}$ for some $k' \in V(G)$.

If there exists a neighbour k of k', such that $\{k, k'\}$ is non-dominating, then the clique $K = \{k, k'\}$ has the desired property. So we may assume that no such k exists and every $k \in N(k')$ is complete to $V(G) \setminus N(k')$. Since for $a, b \in N(k')$ and $c \in V(G) \setminus N(k')$, k'-a-c-b-k' is not a hole of length four, it follows that $N(k')$ is a clique. By the minimality of $V(G)$, there is a bisimplicial vertex v in $G \setminus N(k')$. But now, since $N_G(v) = N_{G \setminus N(k')}(v) \cup N(k')$, it follows that v is bisimplicial in G, a contradiction. This proves 2.4. ▮

In the next few sections, we will be proving several statements about an even-hole-free graph G such that 1.2 holds for all graphs with fewer vertices than G. We refer to this property as "the minimality of $|V(G)|$".

3 Non-dominating holes

The goal of this section is to prove the following:

3.1 *Let G be an even-hole-free graph such that 1.2 holds for all graphs with fewer vertices that G. Let H be a non-dominating hole of G. Then there is a vertex in $V(G) \setminus N(H)$ which is bisimplicial in G.*

Proof. Assume no such vertex exists. Let h_1-$\dots$-h_k-h_1 be the vertices of H in order. Let $M = V(G) \setminus N(H)$ and $N = N(H) \setminus V(H)$. Then no vertex of M is bisimplicial in G. From the minimality of $|V(G)|$ it follows that G is connected, and therefore $N \neq \emptyset$.

(1) M is connected, and every vertex of N has a neighbour in M.

Assume that either M is not connected or there is a vertex $n \in N$ with a non-neighbour in M. In the first case let X be a component of M, and in the second let $X = \{n\}$. Then $M \neq X$, and H is a non-dominating hole in $G \setminus X$. By the minimality of $|V(G)|$, there is a vertex v in $M \setminus X$ that is bisimplicial in $G \setminus X$. But $N_G(v) = N_{G\setminus X}(v)$, and so v is bisimplicial in G, a contradiction. This proves (1).

Let X be the set of leaves, Y the set of hats, C the set of clones, and B the set of major vertices and pyramids with respect to H. By 2.2 $N = X \cup Y \cup C \cup B$.

(2) B is a clique.

Suppose not. Let $b_1, b_2 \in B$ be non-adjacent. By (1), both b_1 and b_2 have neighbours in M and M is connected, so there exists a path P_0 joining b_1 and b_2 and otherwise contained in M.

Assume first that there is a vertex $h \in V(H)$, adjacent to both b_1 and b_2. Let h' and h'' be neighbours of h in H. Since b_1, b_2 are in B, each of them has a neighbour in $V(H) \setminus \{h, h', h''\}$, and therefore there is a path P_1 joining b_1 and b_2 and otherwise contained in $V(H) \setminus \{h, h', h''\}$. But now the paths b_1-P_0-b_2, b_1-h-b_2, b_1-P_1-b_2 form a theta, contrary to 2.1. This proves that no vertex of H is a common neighbour of b_1 and b_2.

We may assume that b_1 is adjacent to h_1. Let i be maximum and j minimum such that b_2 is adjacent to h_i and h_j. Then $i, j \neq 1$. Since b_2 is in B, it follows that $i - j \geq 3$. Let R be the path of H between h_i and h_j containing h_1. Let $h_{i'}$ be the neighbour of b_1 in $V(R)$ such that the subpath P_i of R between h_i and $h_{i'}$ contains no other neighbour of b_1, and let $h_{j'}$ and P_j be defined similarly. If $h_{i'}$ and $h_{j'}$ are distinct and non-adjacent, then b_1-P_0-b_2, b_1-$h_{i'}$-P_i-h_i-b_2, b_1-$h_{j'}$-P_j-h_j-b_2 form a theta in G, contrary to 2.1, so we may assume not, and therefore b_1 has at most two neighbours in $V(R)$.

Assume first that b_1 has exactly two neighbours in $V(R)$, and therefore $h_{i'}$ is non-adjacent to h_j, and $h_{j'}$ to h_i. Since $|i - j| \geq 3$, it follows that b_1 has a neighbour in $V(H) \setminus V(R)$ non-adjacent to one of h_i, h_j, say h_i. So there exists a path Q joining b_1 and b_2 and otherwise contained in $\{h_j, h_{j+1}, \dots, h_{i-2}\}$. But then b_1-P_0-b_2, b_1-Q-b_2, b_1-$h_{i'}$-P_i-h_i-b_2 form a theta in G, contrary to 2.1. This proves that h_1 is the unique neighbour of b_1 in $V(R)$.

From the symmetry and since h_i-R-h_j-b_2-h_i is not a hole of length four, we may assume that $j > 2$. Since b_1 is in B, it follows that b_1 has at least two neighbours in $V(H) \setminus V(R)$, and in particular b_1 has a neighbour in $V(H) \setminus V(R)$ non-adjacent to h_i. So there exists a path Q joining b_1 and b_2 and otherwise contained in $\{h_j, h_{j+1}, \dots, h_{i-2}\}$. But then b_1-P_0-b_2, b_1-Q-b_2, b_1-$h_{i'}$-P_i-h_i-b_2 form a theta in G, contrary to 2.1. This proves (2).

(3) If $b \in B$ and $c \in C$ are non-adjacent and c is a clone at h, then b is a pyramid with apex h.

We may assume that $h = h_1$. Let H' be the hole $H \cup \{c\} \setminus \{h_1\}$. Assume first that b is adjacent to h_1. Then the number of neighbours of b in $V(H')$ differs by one from the number of neighbours of b in $V(H)$, and since G contains no even wheel, b has exactly two neighbours, h_i, h_j in $V(H')$. 2.2 applied to H' implies that h_i is adjacent to h_j. But then b is a pyramid with apex h_1, and (3) holds.

So we may assume that b is non-adjacent to h_1. Let i be maximum and j minimum such that b is adjacent to h_i and h_j. Since b is in B, $i - j \geq 3$. Let P_i, P_j be the subpaths of $H \setminus \{h_1\}$ between h_i and h_k, and h_2 and h_j respectively. By (1), there is a path P_0 joining b and c and otherwise contained in M. But now b-P_0-c, b-h_i-P_i-h_k-c, b-h_j-P_j-h_2-c form a theta in G, contrary to 2.1. This proves (3).

A vertex h of H is a 1-*base* if some vertex of N is either a leaf at h or a clone at h. An edge hh' of H is a 2-*base* if some vertex of N is a hat at hh'.

(4) The set of all 1-bases is a clique.

Suppose not. We may assume that h_1 is a 1-base, and there exists $3 \leq i \leq k-1$ such that h_i is a 1-base. Let x be a leaf or a clone at h_1 and y a leaf or a clone at h_i. By (1), there is a path P_0 joining x and y and otherwise contained in M. Let P_1 and P_2 be the subpaths of $H \setminus \{h_1\}$ joining h_2 and h_{i-1}, and h_{i+1} and h_k, respectively.

Now if x, y are both leaves, then h_1-x-P_0-y-h_i, h_1-h_2-P_1-h_{i-1}-h_i, h_1-h_k-P_2-h_{i+1}-h_i form a theta; if x, y are both clones and x is non-adjacent to y, then x-P_0-y, x-h_2-P_1-h_{i-1}-y, x-h_k-P_2-h_{i+1}-y form a theta; and if, say, x is a leaf and y is a clone, then h_1-x-P_0-y, h_1-h_2-P_1-h_{i-1}-y, h_1-h_k-P_2-h_{i+1}-y form a theta, in all cases a contradiction to 2.1. So x and y are both clones and they are adjacent. But then the graph $G|(V(H) \cup \{x, y\} \setminus \{h_1\})$ is an even wheel, again contrary to 2.1. This proves (4).

(5) At most one edge of H is a 2-base.

Suppose not. We may assume that h_1h_2 is a 2-base, and for some $2 \leq i \leq k-1$, the edge h_ih_{i+1} is a 2-base. Let x be a hat at h_1h_2 and y a hat at h_ih_{i+1}. By (1), there is a path P_0 joining x and y and otherwise contained in M.

Assume first that $i = 2$. Let P be the path $H \setminus \{h_2\}$. Then h_1-P-h_3-y-P_0-x-h_1 is a hole H', and the neighbours of h_2 in H' are precisely $\{h_1, x, y, h_3\}$. So $V(H') \cup \{h_2\}$ induces an even wheel in G, contrary to 2.1. This proves that $i \neq 2$.

Let P_1 be the subpath of $H \setminus \{h_2\}$ joining h_1 and h_{i+1}, and P_2 be the subpath of $H \setminus \{h_1\}$ joining h_2 and h_i. Then the three paths h_1-P_1-h_{i+1}, h_2-P_2-h_i, x-P_0-y form a prism in G, contrary to 2.1. This proves (5).

(6) There does not exist a clique K with $|K| \leq 2$, such that $N \subseteq N(K)$ and $M \not\subseteq N(K)$.

Suppose such K exists. Let $G' = G \setminus (V(H) \setminus K)$. Then K is a clique of size at most two in G', and, since $M \not\subseteq N(K)$, it is non-dominating in G'. It follows from the minimality of $|V(G)|$ that there exists a vertex v in $V(G') \setminus N(K)$ which is bisimplicial in G'. Since $N \subseteq N(K)$, we deduce that $v \in M$. But since $V(G) \setminus V(G') \subseteq V(H)$, it follows that $N_G(v) = N_{G'}(v))$, and therefore v is bisimplicial in G, a contradiction. This proves (6).

(7) Every 1-base is complete to $B \cup C$.

Suppose not. Let b in $B \cup C$ be non-adjacent to a 1-base, say h_1. Let x be a clone or a leaf at h_1. If b belongs to C, we get a contradiction to (4), so we may assume that b is in B.

Assume first that x is a clone, and let H' be the hole with vertex set $V(H) \cup \{x\} \setminus \{h_1\}$. By (3) b is adjacent to x. But now $N(b) \cap V(H') = N(b) \cap V(H) \cup \{x\}$ and so b has an even number, and at least four, neighbours in $V(H')$, contrary to 2.1.

So x is a leaf. Let i be maximum and j minimum such that b is adjacent to h_i and h_j. Let P_i be the subpath of $H \setminus \{h_1\}$ joining h_k and h_i and let P_j be defined similarly. Since b belongs to B, $i - j > 3$. By (1) there exists a path P_0 joining x and b and otherwise contained in M. But now h_1-x-P_0-b, h_1-P_i-h_i-b, h_1-P_j-h_j-b is a theta in G, contrary to 2.1. This proves (7).

(8) C is a clique.

Suppose not, and choose non-adjacent $c_1, c_2 \in C$. We may assume that c_1 is a clone at h_1. Since h_2-c_1-h_k-c_2-h_2 is not a hole of length four in G, it follows that c_2 is not a clone at h_1. By (4) we may assume that c_2 is a clone at h_2; and h_1 and h_2 are the only 1-bases.

First we claim that every 2-base is incident with one of h_1, h_2. Suppose not and let $h_i h_{i+1}$ be a 2-base with $i \neq 1, 2, k$. Let x be a hat at $h_i h_{i+1}$. By (1) all of x, c_1, c_2 have neighbours in M and M is connected. Let $P_0 = p_1$-$\ldots$-p_m be a path with $p_1 = x$, $V(P) \setminus \{p_1\} \subseteq M$, and such that p_m has a neighbour in $\{c_1, c_2\}$ and $\{c_1, c_2\}$ is anticomplete to $P \setminus \{p_m\}$. From the symmetry we may assume that p_m is adjacent to c_2. Since c_1-h_2-c_2-p_m-c_1 is not a hole of length four, c_1 is non-adjacent to p_m, and therefore has no neighbour in $V(P_0)$.

Let P_1 be the subpath of $H \setminus \{h_2\}$ joining h_3 and h_i. Let P_2 be the subpath of $H \setminus \{h_1\}$ joining h_k and h_{i+1}. If $i \neq 3$ then the three paths c_2-p_m-P_0-x, h_3-P_1-h_i, h_2-c_1-h_k-P_2-h_{i+1} form a prism in G; and if $i = 3$ then h_4-P_2-h_k-c_1-h_2-c_2-p_m-P_0-x-h_4 is a hole in G and h_3 has exactly four neighbours in it, in both cases contrary to 2.1. This proves that every 2-base is incident with one of h_1, h_2.

Let $K = \{h_1, h_2\}$. Since h_1, h_2 are the only 1-bases in H, every 2-base is incident with one of h_1, h_2, every vertex of B is adjacent to both of h_1, h_2 by (7), and $N = X \cup C \cup Y \cup B$, it follows that N is included in $N(K)$. But $M \cap N(K) = \emptyset$, contrary to (6). This proves (8).

(9) There exists either a vertex in N that is not complete to M, or a 1-base, or a 2-base.

Suppose not. Then $N = B$ and B is complete to M. It follows from the minimality of $|V(G)|$ that some vertex v of M is bisimplicial in $G|M$. Since $N_G(v) = N_{G|M}(v) \cup B$ and by (2) B is a clique, it follows that v is bisimplicial in G, a contradiction. This proves (9).

(10) There exists a 2-base.

Suppose not, so $Y = \emptyset$. If there exists a 1-base, let K be the set of all 1-bases, and otherwise let $K = \{n\}$ for some $n \in N$ that is not complete to M (the existence of such a vertex n follows from (9)). Then $M \not\subseteq N(K)$. By (4) K is a clique of size at most two. But by (2) and (7), and since $N = B \cup C \cup X$, it follows that $N \subseteq N(K)$, contrary to (6). This proves (10).

In view of (10) we may assume without loss of generality that h_1h_2 is a 2-base.

(11) None of h_1, h_2 is a 1-base.

Suppose one of h_1, h_2 is a 1-base, and from the symmetry we may assume it is h_1. Let K be the set of all 1-bases, then by (4) K is a clique of size at most two. Since by (5) Y is complete to h_1, it follows from (7) that $N \subseteq N(K)$. But now, since $K \subseteq V(H)$, it follows that $N(K) \cap M = \emptyset$, contrary to (6). This proves (11).

For a vertex v in $B \cup C$ let $i(v)$ the minimum $i > 2$ such that v is adjacent to h_i. We say that v is of *even type* if $i(v)$ is even, and of *odd type* otherwise. Let T be the set of all vertices of even type. Please note that T is anticomplete to h_2.

(12) $B \cup C$ is a clique.

Suppose not. It follows from (2), (3) and (8) that there exist a vertex h_j of H, a clone c at h_j, and a pyramid p with apex h_j such that c is non-adjacent to p. By (11), $j \neq 1, 2$. Let h_ih_{i+1} be the base of p.

First we claim that h_j is the only 1-base in H. For suppose for some $m \neq j$, h_m is another 1-base. By (4) $m \in \{j-1, j+1\}$, and by (7) p is adjacent to h_m, contrary to the fact that h_j is the apex of p. This proves the claim.

Next we claim that $i = 1$. Suppose not. From the symmetry we may assume that $j < i$. Let x be a hat at h_1h_2. By (1) all of x, c, p have neighbours in M and M is connected. Let $P_0 = p_1\text{-}\dots\text{-}p_m$ be a path with $p_1 = x$, $V(P_0) \setminus \{p_1\} \subseteq M$, and such that p_m has a neighbour in $\{c, p\}$ and $\{c, p\}$ is anticomplete to $P_0 \setminus \{p_m\}$. Since $c\text{-}h_j\text{-}p\text{-}p_m\text{-}c$ is not a hole of length four, not both c and p are adjacent to p_m, and therefore one of c, p has no neighbour in $V(P_0)$.

If p is adjacent to p_m, then the subgraph induced by G on $V(H) \cup V(P_0) \cup \{p, c\} \setminus \{h_j\}$ is an even wheel if $i = k$ and a prism if $i \neq k$, contrary to 2.1. If c is adjacent to p_m, let P_1 be the subpath of $H \setminus \{h_i\}$ between h_{i+1} and h_1, and let P_2 be the subpath of $H \setminus \{h_1\}$ between h_2 and h_{j-1}. Then the three paths $c\text{-}p_m\text{-}P_0\text{-}x$, $h_j\text{-}p\text{-}h_{i+1}\text{-}P_1\text{-}h_1$, $h_{j-1}\text{-}P_2\text{-}h_2$ form a prism if $j > 3$ and an even wheel otherwise, contrary to 2.1. This proves that $i = 1$. Consequently $4 \leq j \leq k-1$.

Let $L = \{h_3, h_4, \dots, h_{j-1}\}$ and let B' be the set of all vertices of B that are anticomplete to L. Let $S = \{h_2\}$, $K = B \cup C \cup \{h_j\} \setminus B'$ and $R = M \cup X \cup Y \cup B'$. Then $G|(K \cup S \cup R \cup L)$ is even-hole-free, by (3) and (7) both K and S are cliques, and L is connected and every vertex of K has a neighbour in L. Let G' be the graph obtained from $G \setminus (V(H) \setminus \{h_2, h_j\})$ by adding edges between h_2 and all its non-neighbours in $T \cup \{h_j\}$. By 2.3 applied to $G|(K \cup S \cup L \cup R)$, it follows that G' is even-hole-free.

Let $U = \{h_2, h_j\}$. Then M is disjoint from $N_{G'}(U)$, and therefore U is a non-dominating clique in G'. By (7) $B \cup C \cup X$ is complete to h_j and Y is complete to h_2, so N is included in $N_{G'}(U)$. It follows from the minimality of $|V(G)|$ that there is a vertex $v \in V(G') \setminus N_{G'}(U)$ that is bisimplicial in G', and therefore v is in M. Since $V(G) \setminus V(G') \subseteq V(H)$, we deduce that $N_G(v) = N_{G'}(v)$, and so v is bisimplicial in G, a contradiction. This proves (12).

(13) $X \cup T \neq \emptyset$.

Suppose $X \cup T$ is empty. Then $N = B \cup Y \cup C$. Let $S = \{h_3\}$, $K = B \cup C$, $L = H \setminus \{h_1, h_2, h_3\}$ and $R = M \cup Y \cup \{h_2\}$. Then K, S are both cliques, L is connected and anticomplete to R, and every vertex of K has a neighbour in L. Let G' be the graph obtained from $G|(K \cup S \cup R)$ by adding all edges between $B \cup C$ and h_3. It follows from 2.3 that G' is even-hole-free.

Let $U = \{h_2, h_3\}$. Since $N(U) \cap M = \emptyset$, U is a non-dominating clique in G'. By the minimality of $|V(G)|$, it follows that there is a bisimplicial vertex v in $V(G') \setminus N(U)$. Since $V(G') = \{h_2, h_3\} \cup B \cup C \cup Y \cup M$, and Y is complete to h_2 and $B \cup C$ to h_3, we deduce that v is in M. But now $N_G(v) = N_{G'}(v)$, and therefore v is bisimplicial in G, a contradiction. This proves (13).

(14) For some even integer i with $3 < i < k$ there is a leaf at h_i.

Suppose not. Let K' be the set of all vertices h_j of H such that there is a leaf at h_j. By (4) and (11) $|K'| \leq 1$.

Assume first that K' is empty and T is complete to M. Then $N = B \cup C \cup Y$. Since every $y \in Y$ has a neighbour $m \in M$, every $t \in T$ is adjacent to m, and y-h_2-$\dots$-$h_{i(t)}$-t-m-y is not an even hole in G, it follows that Y is complete to T. Choose $t \in T$ (by (13) $T \neq \emptyset$). Since H is a non-dominating hole in $G \setminus \{t\}$, it follows from the minimality of $|V(G)|$ that some vertex m of M is bisimplicial in $G \setminus \{t\}$. Now t is complete to $N_{G \setminus \{t\}}(m)$, because $N_{G \setminus \{t\}}(m) \subseteq M \cup Y \cup B \cup C$, and by (12) and the previous argument t is complete to $M \cup Y \cup B \cup C$. Since $N_G(m) = N_{G \setminus \{t\}}(m) \cup \{t\}$, it follows that m is bisimplicial in G, a contradiction. This proves that either $K' \neq \emptyset$ or some vertex of T is not complete to M.

Let L be the subpath of $H \setminus \{h_1\}$ from h_3 to h_{j-1} if $K' = \{h_j\}$, and let L be the path $H \setminus \{h_1, h_2\}$ if $K' = \emptyset$. Let $S = \{h_2\}$, and define K to be the union of K' with the set of all the vertices of $B \cup C$ that have a neighbour in L. Let $R = M \cup X \cup Y \cup B \cup C \setminus S$.

Then by (7) and (12) both K, S are cliques, L is connected and anticomplete to R and every vertex of K has a neighbour in L. Let G' be the graph obtained from $G|(R \cup K \cup S)$ by adding all edges between $K' \cup T$ and h_2. By 2.3 G' is even-hole-free. By (13) $K' \cup T$ is non-empty. If $K' = \{h_j\}$ let $a = h_j$ and otherwise let a be a vertex in T that is not complete to M. Let $U = \{a, h_2\}$. Then U is a non-dominating clique of size two in G', and it follows from the minimality of $|V(G)|$ that there is a vertex $v \in V(G') \setminus N_{G'}(U)$ that is bisimplicial in G'. Since, by (7) and (12), $B \cup C \cup X$ is complete to a and, by (5), Y is complete to h_2, it follows that v is in M. But then, since $V(G) \setminus V(G')$ is included in $V(H)$, it follows that $N_G(v) = N_{G'}(v)$, and so v is bisimplicial in G, a contradiction. This proves (14).

In view of (14) let i_0 be an even integer such that $3 < i_0 < k$ and h_{i_0} is a 1-base. By (4), the set of all 1-bases is a clique included in $\{h_{i_0-1}, h_{i_0}, h_{i_0+1}\}$, and from the symmetry we may assume that h_{i_0-1} is not a 1-base. Let L be the subpath of $H \setminus \{h_1\}$ between h_2 and h_{i_0-1}, and let R be the union of $X \cup M \cup \{h_{i_0+1}\}$ with the set of all vertices of $B \cup C$ that have no neighbour in L. Let $K = \{h_{i_0}\} \cup B \cup C \setminus R$ and let $S = Y$. Then (by (7) and (12)) K is a clique, L is connected, anticomplete to R, and every vertex of $K \cup S$ has a neighbour in L. Moreover, h_2 is a vertex of L complete to S, and S is anticomplete to $L \setminus \{h_2\}$. Let G' be the graph obtained from $G|(K \cup S \cup R)$ by adding all edges between $K \setminus T$ and Y. By 2.3 G' is even-hole-free. Let $U = \{h_{i_0}, h_{i_0+1}\}$. Then U is a clique of size two in G', and since M is anticomplete to U, it is non-dominating. It follows from the minimality of $|V(G)|$ that some vertex v of $V(G') \setminus N_{G'}(U)$ is bisimplicial. Since U contains all 1-bases, by (7) $B \cup C$ is complete to h_{i_0} and Y is complete to h_{i_0} by the construction of G', it follows that v belongs to M. But since $V(G) \setminus V(G')$ is a subset of $V(H)$, it follows that $N_G(v) = N_{G'}(v)$, and so v is bisimplicial in G, a contradiction. This completes the proof of 3.1. ∎

4 Star Cutsets

A *cutset* in G is a subset C of $V(G)$ such that $V(G) \setminus C$ is the union of two non-empty sets, anticomplete to each other. A *star cutset* is a cutset consisting of a vertex and some of its neighbours. If v together with a subset of $N(v)$ is a cutset, we say that v is a *centre* of this star cutset. A star cutset C is called *full* if it consists of a vertex and all its neighbours. A *double star cutset* in G is a cutset consisting of two adjacent vertices u, v and some of their neighbours. The edge uv is then a *centre* of the double star cutset.

In the next few theorems, we develop tools that allow us to make use of certain variations of star cutsets and double star cutsets in the proof of 1.2.

4.1 *Let G be an even-hole-free graph such that 1.2 is true for all graphs with fewer vertices than G. Assume that there exists a non-dominating clique K of size at most two in G such that no vertex of $V(G) \setminus N(K)$ is bisimplicial in G. Let C be a star cutset of G with centre c such that some component of $V(G) \setminus C$ is disjoint from K and is not complete to c. Then $K \subseteq C \setminus \{c\}$.*

Proof. By 2.4, we may assume that $|K| = 2$. Let $K = \{x, y\}$. Let $C_1, \ldots, C_k$ be the components of $V(G) \setminus C$. Then $k \geq 2$. Let $G_i = G|(C \cup C_i)$.

(1) $c \notin K$.

Suppose $c \in K$. Assume without loss of generality that $c = x$. Since K is a non-dominating clique in G, $K \cap V(G_i)$ is non-dominating in G_i for some $1 \leq i \leq k$. From the minimality of $|V(G)|$, there exists a vertex $v \in V(G_i) \setminus N(K)$, such that v is bisimplicial in G_i. It follows that $v \in C_i$. But then $N_{G_i}(v) = N_G(v)$, and so v is bisimplicial in G, a contradiction. This proves (1).

(2) $K \cap C \neq \emptyset$.

Suppose $K \cap C = \emptyset$. Then we may assume that $K \subseteq C_1$. By hypothesis some other component of $V(G) \setminus C$, say C_2, is not complete to c. Thus $\{c\}$ is a non-dominating clique in G_2. By the minimality of $|V(G)|$, there exists a vertex $v \in V(G_2) \setminus N(\{c\})$, such that v is bisimplicial in G_2. It follows that $v \in C_2$. But now $N_{G_2}(v) = N_G(v)$, and so v is bisimplicial in G, a contradiction. This proves (2).

To complete the proof, suppose that $K \not\subseteq C$. By (1) and (2), we may assume that $x \in C \setminus \{c\}$, and $y \in C_1$. For $2 \leq i \leq k$, let $C'_i = C_i \setminus N(c)$ and let $C''_i = C_i \cap N(c)$. Since some component of $V(G) \setminus C$ is disjoint from K and is not complete to c, it follows that $\bigcup_{i=2}^k C'_i \neq \emptyset$. Assume first that for some $2 \leq i \leq k$, x is not complete to C'_i. Then $\{c, x\}$ is a non-dominating clique in G_i, and by the minimality of $|V(G)|$, there exists a vertex $v \in V(G_i) \setminus N(\{c, x\})$, such that v is bisimplicial in G_i. It follows that $v \in C'_i$. But then $N_{G_i}(v) = N_G(v)$, and so v is bisimplicial in G, a contradiction. This proves that x is complete to $\bigcup_{i=2}^k C'_i$.

We claim that for every $2 \leq i \leq k$, x is complete to $(C \cup C''_i \setminus \{x\}) \cap N(C'_i)$. For suppose not, choose $n \in (C \cup C''_i \setminus \{x\}) \cap N(C'_i)$ non-adjacent to x, and let $c_1 \in C'_1$ be a neighbour of n. Then x is adjacent to c_1, and n-c_1-x-c-n is a hole of length four, a contradiction. This proves the claim.

Let $G' = G|(C \cup C_1 \cup \bigcup_{i=2}^k C''_i)$. Since K is a non-dominating clique in G, and x is complete to $\bigcup_{i=2}^k C'_i$, it follows that K is a non-dominating clique in G'. By the minimality of $|V(G)|$, there exists a vertex $v \in V(G') \setminus N(K)$, such that v is bisimplicial in G'. Since x is adjacent to c, and for every $2 \leq i \leq k$, x is complete to $(C \cup C''_i) \cap N(C'_i)$, it follows that either v belongs to $\bigcup_{i=2}^k C''_i \cup C \setminus \{c\}$, and v is anticomplete to $V(G) \setminus V(G')$, or v belongs to C_1. In both cases, $N_{G'}(v) = N_G(v)$, and so v is bisimplicial in G, a contradiction. This proves 4.1. ∎

4.2 *Let G be an even-hole-free graph such that 1.2 is true for all graphs with fewer vertices than G. Assume that there exists a non-dominating clique K of size at most two in G such that no vertex of $V(G) \setminus N(K)$ is bisimplicial in G. Then G does not admit a full star cutset.*

Proof. By 2.4, we may assume that $|K| = 2$; and let the vertices of K be x and y. Suppose there exists $w \in V(G)$ such that $N(w)$ is a cutset in G. Let $N = N(w) \setminus \{w\}$ and let $C_1, \dots, C_k$ be the components of $V(G) \setminus N(w)$. Then $k \geq 2$. Let $G_i = G|(C_i \cup N(w))$. By 4.1, $K \subseteq N$.

(1) $C_i \setminus N(K) \neq \emptyset$ for every $1 \leq i \leq k$.

Suppose $C_1 \in N(K)$. Since K is a non-dominating clique in G, it follows that K is a non-dominating clique in $G' = G \setminus C_1$. By the minimality of $|V(G)|$, there exists a vertex $v \in V(G') \setminus N(K)$, such that v is bisimplicial in G'. If either $v \in \bigcup_{i=2}^k C_i$, or $v \in N$ and v is anticomplete to C_1, then $N_{G'}(v) = N_G(v)$, and so v is bisimplicial in G, a contradiction. So $v \in N$, and v has a neighbour $c \in C_1$. Without loss of generality we may assume that c is adjacent to x. But now v-c-x-w-v is a hole of length four, a contradiction. This proves (1).

(2) $C_i \cap N(x) \neq \emptyset$ and $C_i \cap N(y) \neq \emptyset$ for every $1 \leq i \leq k$.

Suppose $C_1 \cap N(x) = \emptyset$. By (1), y is not complete to C_1, and therefore $\{w, y\}$ is a non-dominating clique in G_1. By the minimality of $|V(G)|$, there exists a vertex $v \in V(G_1) \setminus N(\{w, y\})$, such that v is bisimplicial in G_1. But now, $v \in C_1$ and so $N_{G_1}(v) = N_G(v)$ and $v \notin N(K)$. Consequently, v is bisimplicial in G, and $v \in V(G) \setminus N(K)$, a contradiction. This proves (2).

Let W be the set of vertices in $N(w) \setminus N(K)$ that are anticomplete to $\bigcup_{i=1}^k C_i$. Let $Z = N(w) \setminus (N(K) \cup W)$ and let $Z_i = N(C_i) \cap Z$.

(3) For $1 \leq i < j \leq k$, $Z_i \cap Z_j = \emptyset$ and there does not exist a path from a vertex of Z_i to a vertex of Z_j with interior in W.

Suppose (3) is false. Assume first that there exist $z_1 \in Z_1$ and $z_2 \in Z_2$, such that there is a path R of even length from z_1 to z_2 with $R^* \subseteq W$, and either $z_1 = z_2$, or $z_1 \notin Z_2$ and $z_2 \notin Z_1$. By (2), x has a neighbour in C_1 and in C_2, and so for $m = 1, 2$ there exists a path P_m between x and z_m, such that $P_m^* \subseteq C_m$. Since x-P_m-z_m-w-x is not an even hole, P_m is odd for $m = 1, 2$. But now, x-P_1-z_1-R-z_2-P_2-x is an even hole in G, a contradiction. This proves that for $1 \leq i < j \leq k$, $Z_i \cap Z_j = \emptyset$ and every path from a vertex of Z_i to a vertex of Z_j with interior in W is odd.

We may therefore assume that there exist $z_1 \in Z_1$, $z_2 \in Z_2$, and a path R of odd length from z_1 to z_2 with $R^* \subseteq W$. Suppose there exist a path P_1 from x to z_1 with $P_1^* \subseteq C_1 \setminus N(y)$ and a path P_2 from y to z_2 with $P_2^* \subseteq C_2 \setminus N(x)$. Since x-P_1-z_1-w-x and y-P_2-z_2-w-y are not even holes, it follows that both P_1 and P_2 are odd. But then x-P_1-z_1-R-z_2-P_2-y-x is an even hole, a contradiction. This proves that no such P_1 and P_2 exist.

From the symmetry assume that there is no path from z_1 to x with interior in $C_1 \setminus N(y)$. Let S be the union of the components of $C_1 \setminus N(y)$ that contain no neighbour of x. Now z_1 has a neighbour, say c, in C_1. Since z_1-c-y-w-z_1 is not a hole of length four, it follows that y is non-adjacent to c. Since there is no path from z_1 to x with interior in $C_1 \setminus N(y)$, it follows that $c \in S$. Let $G' = G|(N(w) \cup (N(y) \cap C_1) \cup S)$. Then $\{w, y\}$ is a non-dominating clique in G'. By the minimality of $|V(G)|$, there exists a vertex $v \in V(G') \setminus N(\{w, y\})$, such that v is bisimplicial in G'. But this means that $v \in S$, and therefore v is anticomplete to $\{x, y\}$ and $N_{G'}(v) = N_G(v)$; and so v is bisimplicial in G, a contradiction. This proves (3).

For $1 \leq i \leq k$, let W_i be the set of vertices $a \in W$, such that there is a path from w to a vertex of Z_i, with interior in W, and let $W_0 = W \setminus \bigcup_{i=1}^{k} W_i$. By (3), W_i and W_j are disjoint and anticomplete to each other for $0 \leq i < j \leq k$. Let $G' = G \setminus \bigcup_{i=2}^{k} (C_i \cup Z_i \cup W_i)$. By (1), K is a non-dominating clique in G'. By the minimality of $|V(G)|$, there exists a vertex $v \in V(G') \setminus N(K)$, such that v is bisimplicial in G'. But this means that $v \in C_1 \cup Z_1 \cup W_1 \cup W_0$, and therefore $N_{G'}(v) = N_G(v)$; and so v is bisimplicial in G, a contradiction. This completes the proof of 4.2. ∎

4.3 *Let G be an even-hole-free graph such that 1.2 is true for all graphs with fewer vertices than G. Assume that there exists a non-dominating clique K of size at most two in G such that no vertex of $V(G) \setminus N(K)$ is bisimplicial in G. Then there does not exist a double star cutset C in G with centre uv and a vertex $w \in V(G) \setminus (N(u) \cup N(v))$, such that K and w are contained in two different components of $V(G) \setminus C$.*

Proof. Let C_1 be the component of $V(G) \setminus C$ with $w \in C_1$, and let $G' = G|(C \cup C_1)$. Then $\{u, v\}$ is a non-dominating clique of size two in G'. By the minimality of $|V(G)|$, there exists a vertex $v \in V(G') \setminus N(\{u, v\})$, such that v is bisimplicial in G'. But this means that $v \in C_1$, and therefore v is anticomplete to K and $N_{G'}(v) = N_G(v)$, and so v is bisimplicial in G, a contradiction. This proves 4.3. ∎

Another useful fact of similar flavor is the following:

4.4 *Let G be an even-hole-free graph such that 1.2 is true for all graphs with fewer vertices than G. Assume that there exists a non-dominating clique $K = \{x, y\}$ in G such that no vertex of $V(G)\setminus N(K)$ is bisimplicial in G. Let H be a hole in G, such that $x \in V(H)$ and $y \notin V(H)$. Then $G\setminus(N(H)\setminus\{y\})$ is connected.*

Proof. Suppose $G\setminus(N(H)\setminus\{y\})$ is not connected, and let C_1 be the component of $G\setminus(N(H)\setminus\{y\})$ containing y, and $C_2 \neq \emptyset$ some other component. Let $G' = G|(C_2 \cup N(H) \setminus \{y\})$. Since $C_2 \neq \emptyset$, H is a non-dominating hole in G'. By the minimality of $|V(G)|$, there exists a vertex $v \in V(G') \setminus N(H)$ such that v is bisimplicial in G'. But this means that $v \in C_2$, and therefore v is anticomplete to $\{x, y\}$ and $N_G(v) = N_{G'}(v)$; so v is bisimplicial in G, a contradiction. This proves 4.4. ∎

Let H be a hole, and w a major vertex with respect to H that is not complete to $V(H)$. Let us call a *w-interval* a maximal path of H whose vertex set is complete to w, and a *w-gap* a maximal path of H whose vertex set is anticomplete to w. Thus every vertex of H either belongs to a unique w-gap or to a unique w-interval. For a w-gap C, let the *borders* of C be the ends of the path $H \setminus V(C)$. So the borders of a gap are adjacent to w.

In view of 4.2 and 4.1, it is of interest to us to find out which even-hole-free graphs admit a star cutset. While we do not know the complete answer to this question, we can prove the following:

4.5 *Let G be even-hole-free, and let H be a hole in G, such that some vertex w of G is major with respect to H. Assume that w is not complete to $V(H)$. Then G admits a star cutset with centre w. Moreover, let C be a w-gap of H with borders x, y. Let A be the set of all vertices h in $V(H) \cap N(w)$ such that the subpath of $H \setminus \{x\}$ from y to h contains an even number of neighbours of w, and let $B = V(H) \setminus (V(C) \cup N(w))$. Let $N' = N(w) \setminus A$. Then $V(G) \setminus (N' \cup \{w\})$ is the union of two sets V_1 and V_2, such that V_1 is anticomplete to V_2, $V(C) \subseteq V_1$ and $A \cup B \subseteq V_2$.*

We start with some lemmas.

4.6 *Let G be a graph and let P be a path in G with vertices $p_1, \ldots, p_n$ in order, and let x, y be two non-adjacent vertices in $V(G) \setminus V(P)$ such that each of x, y has two non-adjacent neighbours in $V(P)$. Suppose there do not exist two paths S_1 and S_2 between x and y such that $S_1^* \cup S_2^* \subseteq V(P)$ and S_1^* is anticomplete to S_2^*. Then, possibly with x and y exchanged, there exists $1 \leq i \leq n$ such that $N(x) \cap V(P) \subseteq \{p_1, \ldots, p_{i+1}\}$ and $N(y) \cap V(P) \subseteq \{p_i, \ldots, p_n\}$.*

Proof. Let i_x be minimum and j_x maximum such that x is adjacent to p_{i_x} and p_{j_x}, and let i_y and j_y be defined similarly for y. We may assume that $i_x \leq i_y$. If $j_x \leq i_y + 1$, then the theorem holds, so we may assume not, and therefore p_{j_x} and p_{i_y} are distinct and non-adjacent. Since y has two non-adjacent neighbours in $V(P)$, $j_y > i_y + 1$, and in particular p_{i_y} and p_{j_y} are distinct and non-adjacent. Let P_1 be the subpath of P between p_{i_x} and p_{i_y} and P_2 the subpath of P between p_{j_x} and p_{j_y}. Then $V(P_1)$ is anticomplete to $V(P_2)$, and there exist paths S_1, S_2 between x and y with $S_1^* \subseteq V(P_1)$ and $S_2^* \subseteq V(P_2)$, a contradiction. This proves 4.6. ∎

We say that two major vertices x and y with respect to a hole H *cross* if there do not exist paths P_1 and P_2 of H with $|V(P_1) \cap V(P_2)| \leq 1$ such that $N(x) \cap V(H) \subseteq V(P_1)$ and $N(y) \cap V(H) \subseteq V(P_2)$.

4.7 *Let G be even-hole free. Let H be a hole and let x, y be major vertices with respect to H. If x and y cross then x is adjacent to y.*

Proof. Suppose not. Since $G|(V(H) \cup \{x\})$ and $G|(V(H) \cup \{y\})$ are not even wheels, it follows that each of x and y has an odd number of neighbours in $V(H)$. Let the vertices of H be h_1-h_2-$\ldots$-h_k-h_1. First we prove the following useful fact.

(1) x and y have no common neighbour in $V(H)$.

Suppose h_1 is adjacent to both x and y. Let $q, r \in \{3, \ldots, k-1\}$ be such that q is minimum and r is maximum with x adjacent to h_q and h_r. Let $s, t \in \{3, \ldots, k-1\}$ be such that s is minimum and t is maximum with y adjacent to h_s and h_t. Since x and y are both major, h_q is different from h_r, and h_s from h_t. Let $t' = k$ if y is adjacent to h_k, and let $t' = t$ otherwise, and define q' similarly.

We claim that $r > s$. Suppose $r \leq s$. Since x and y cross, we may assume, from the symmetry, that x is adjacent to h_k. Since $G|(\{h_r, h_{r+1}, \ldots, h_k, x, y\})$ is not an even wheel or a theta by 2.1, it follows that y is adjacent to h_2. Since x and y are both major, it follows that $r \geq q' + 2$ and $t' \geq s + 2$. But now the three subpaths of $H \setminus \{h_1\}$ from h_2 to $h_{q'}$, from h_r to h_s and from $h_{t'}$ to h_k, together with x and y, form a theta, contrary to 2.1. This proves that $r > s$. Similarly, $t > q$.

Let $A = H \setminus \{h_1, h_2, h_k\}$. Assume first that each of x, y has two non-adjacent neighbours in $V(A)$. If there there exist two paths S_1 and S_2 between x and y such that $S_1^* \cup S_2^* \subseteq V(A)$ and S_1^* is anticomplete to S_2^*, then $G|(V(S_1) \cup V(S_2) \cup \{h_1\})$ is a theta, contrary to 2.1. So no such pair of paths exists, and, by 4.6 applied to x, y and A, and from the symmetry, we may assume that $r = s + 1$.

Since x-h_1-y-h_r-x and x-h_1-y-h_s-x are not holes of length four, x is non-adjacent to h_s and y is non-adjacent to h_r. Let $s'' > s$ be minimum such that y is adjacent to $h_{s''}$. Since y is major, we deduce that $s'' < k$. But now the paths y-h_s-h_r, y-h_1-x-h_r and y-$h_{s''}$-A-h_r form a theta, contrary to 2.1. This proves that not both x and y have two non-adjacent neighbours in $V(A)$, and hence we may assume that $r = q + 1$.

Since x is major and $G|(V(H) \cup \{x\})$ is not an even wheel by 2.1, it follows that x is adjacent to h_k and h_2. Since x-h_k-y-h_2-x is not a hole of length four, y is non-adjacent to at least one of h_k and h_2 , and therefore y has two non-adjacent neighbours in $V(A)$. Since x-h_1-y-h_q-x and x-h_1-y-h_r-x are not holes of length four, it follows that y is non-adjacent to h_q and h_r. Let $t'' > r$ be minimum such that y is adjacent to $h_{t''}$ and let $s'' < q$ be maximum such that y is adjacent to $h_{s''}$. Let $s' = 2$ if y is adjacent to h_2, and let $s' = s$ otherwise. Let $A' = H \setminus \{h_1\}$.

Now $t'' \leq t \leq t'$. If $t'' < t' - 1$, then the three paths y-$h_{t'}$-A'-h_k-x, y-$h_{t''}$-A'-h_r-x, y-$h_{s'}$-A'-h_2-x form a theta; and if $t' = t'' + 1$ then the three paths $h_{t'}$-A'-h_k-x, y-$h_{s''}$-A'-h_q and $h_{t''}$-A'-h_r form a prism, in both cases contrary to 2.1. So $t'' = t = t'$, and from the symmetry $s'' = s = s'$, and therefore the only neighbours of y in $V(H)$ are h_1, h_s and h_t. But now the three paths h_t-A'-h_k-x, h_t-A'-h_r-x, and h_t-y-h_s-A'-h_2-x form a theta, again contrary to 2.1. This proves (1).

To finish the proof, we may assume using (1) that y is adjacent to h_1 and there exists m with $2 \leq m \leq k-1$ such that y-h_1-h_2-$\ldots$-h_m-x is a path. Let $A = H \setminus \{h_k, h_1, \ldots, h_m, h_{m+1}\}$.

Assume that x is adjacent to h_k. Since by (1) x and y have no common neighbour and y-h_{k-1}-h_k-h_1-y is not a hole of length four, y is non-adjacent to h_k, h_{k-1}. Since x, y are major and $G|(V(H) \cup \{x\})$ is not an even wheel by 2.1, it follows that each of x, y has at least one neighbour in $V(A) \setminus \{h_{k-1}\}$. Consequently there is a path P between x and y with interior in $V(A) \setminus \{h_{k-1}\}$. But then the paths h_1-h_k-x, h_1-h_2-H-h_m-x and h_1-y-P-x form a theta, contrary to 2.1. This proves that x is not adjacent to h_k, and similarly y is not adjacent to h_{m+1}.

Let $s > 1$ be minimum such that y is adjacent to h_s. Then $s \geq m+2$. Let P be the subpath of $H \setminus \{h_k\}$ between h_1 and h_s. Choose q maximum with $m \leq q \leq s$ such that x is adjacent to h_q. By (1), $q < s$.

Assume first that $q = m$. Since both x and y are major, they each have a neighbour in $V(H) \setminus (V(P) \cup \{h_k, h_{s+1}\})$, and therefore there exists a path R from x to y, with interior in $V(H) \setminus (V(P) \cup \{h_k, h_{s+1}\})$. But now, the three paths y-h_1-P-h_q, y-h_s-P-h_q and y-R-x-h_q form a theta, contrary to 2.1.

Next assume that $q > m+1$. Since x and y cross, x has a neighbour in $V(H) \setminus V(P)$, and since $G|(V(P) \cup \{x,y\})$ is not an even wheel, x has at least two neighbours in $V(H) \setminus V(P)$. Since x is non-adjacent to h_k, and y is major, it follows that both x and y have a neighbour in $V(H) \setminus (V(P) \cup \{h_k, h_{s+1}\})$, and therefore there exists a path R from x to y, with interior in $V(H) \setminus (V(P) \cup \{h_k, h_{s+1}\})$. But now the three paths y-h_1-P-h_m-x, y-h_s-P-h_q-x and y-R-x form a theta, contrary to 2.1. This proves that $q = m+1$.

From the symmetry, we deduce that y is adjacent to h_k, and there exists $r < k$, such that x-h_r-h_{r+1}-$\ldots$-h_k-y is a path, say Q. Since both x and y are major, it follows that $r > s+1$. But now the paths x-Q-h_k, y-h_s-P-h_{m+1} and h_1-P-h_m form a prism, contrary to 2.1. This completes the proof of 4.7. ∎

Let H be a hole, and w a major vertex with respect to H. A path Q with vertices $q_1, \ldots, q_k$ in order such that $V(Q) \cap V(H) = \emptyset$, is called an *$(H,w)$-pyramid path* if there exist distinct vertices $x, y, z \in V(H)$ such that

- q_1 is adjacent to x, and q_k is adjacent to y and z, and there are no other edges between $\{q_1, \ldots, q_k\}$ and $V(H)$,
- y and z are adjacent,
- w is anticomplete to $\{q_1, \ldots, q_k\}$,
- w is adjacent to x,
- the subpath of $H \setminus \{y\}$, from x to z contains an odd number of neighbours of w, and
- the subpath of $H \setminus \{z\}$ from x to y contains an odd number of neighbours of w.

We call x the *apex* of the pyramid path, and yz the *base*.

4.8 *Let G be even-hole-free, let H be a hole in G and let w be a major vertex with respect to H. Let T be a path of $G \setminus (V(H) \cup \{w\})$ with vertices $t_1, \ldots, t_m$ in order such that there exist distinct vertices $u, u', v \in V(H)$, t_1 is adjacent to u and u', t_m is adjacent to v, there are no other edges between $\{t_1, \ldots, t_m\}$ and $V(H)$, u is adjacent to u', and $\{t_1, \ldots, t_m\}$ is anticomplete to w. Let Q be the path of $H \setminus \{u'\}$ from u to v, and let Q' be the path of $H \setminus \{u\}$ from u' to v. Assume that each of $V(Q)$ and $V(Q')$ contains a neighbour of w. Then T is an (H,w)-pyramid.*

Proof. It is enough to show that w is adjacent to v, and each of $V(Q)$, $V(Q')$ contains an odd number of neighbours of w. Suppose v is non-adjacent to w. Since $G|(V(H) \cup \{w\})$ is not an even wheel by 2.1, w has an odd number of neighbours in $V(H)$, and from the symmetry we may assume that $V(Q)$ contains an even number of neighbours of w. Since $G|(V(Q) \cup V(T) \cup \{w\})$ is not an even wheel by 2.1, w has exactly two neighbours in $V(Q)$, say x and y, and x is adjacent to y, by 2.2. We may assume that the subpath of Q from u to x does not contain y. Let z be a neighbour of w in Q' such that the subpath of Q' from u' to z contains no other neighbours of w. Since w is major with respect to H, it follows that z is non-adjacent to v. But now, if $x \neq u$, the three paths u-Q-x, u'-Q'-z-w and t_1-T-t_m-v-Q-y form a prism, and if $u = x$, then u has exactly four neighbours in the hole y-w-z-Q'-u'-t_1-T-t_m-v-Q-y, in both cases contrary to 2.1. This proves that v is adjacent to to w.

Since by 2.1 w has an odd number of neighbours in $V(H)$, it follows that the parity of the number of neighbours of w in $V(Q)$ and $V(Q')$ is the same. We may assume that w has an even number of neighbours in $V(Q)$ and $V(Q')$, for otherwise the theorem holds. Since neither of $G|(V(Q) \cup V(T) \cup \{w\})$ and $G|(V(Q') \cup V(T) \cup \{w\})$ is an even wheel by 2.1, it follows that w has exactly two neighbours in $V(Q)$ and they are adjacent, by 2.2, and the same holds for $V(Q')$. But now, since w is adjacent to v, we deduce that the neighbours of w in $V(H)$ are v and the two neighbours of v in H, contrary to the fact that w is major. This proves 4.8. ∎

4.9 *Let G be even-hole-free, let H be a hole in G and let w be a major vertex with respect to H. Let $u, v \in V(H)$ be non-adjacent, and let P be a path with vertices $u, p_1, \ldots, p_k, v$ in order such that*

1. *$P^* \cap (V(H) \cup \{w\}) = \emptyset$,*
2. *w is anticomplete to $\{p_1, \ldots, p_k\}$, and*
3. *each of the paths of H between u and v contains a neighbour of w in its interior.*

Then there exist $i, j \in \{1, \ldots, k\}$ such that the subpath of P between p_i and p_j is an (H, w)-pyramid path.

Proof. We use induction on k. We observe that since $G|(V(H) \cup \{w\})$ is not an even wheel, it follows that $|N(w) \cap V(H)|$ is odd.

(1) If $k = 1$ then the theorem holds.

Suppose $k = 1$. Since p_1 is non-adjacent to w, 4.7 implies that p_1 is not a major vertex. Since p_1 has two non-adjacent neighbours in $V(H)$, namely u and v, it follows that p_1 is a pyramid or a clone, and from the symmetry we may assume that the neighbours of p_1 in $V(H)$ are u, u' and v, where u is adjacent to u'. But now the theorem holds by 4.8. This proves (1).

In view of (1) we may assume that $k \geq 2$. Let $N = V(H) \cap N(p_1)$ and let $M = V(H) \cap N(p_k)$.

(2) One of the following holds:

- *N is contained in the union of the vertex set of some non-empty w-gap of H and its borders, and at least one vertex of the gap belongs to N, or*
- *$|N| = 1$ and w is complete to N*

- $|N| = 2$, *w is complete to N, and the two vertices of N are adjacent to each other*

and the same for M.

Suppose there exists a vertex $n \in N \setminus N(w)$, and let C be the w-gap containing n. Let x and y be the borders of C. If N contains a vertex $n' \in V(H) \setminus (V(C) \cup \{x, y\})$, then the path n-p_1-n' contradicts the minimality of k. So no such n' exists and the first outcome of (2) holds. This proves that we may assume that $N \subseteq N(w)$. Now, if N contains two non-adjacent vertices n and n', then n-p_1-n'-w-n is a hole of length four, a contradiction; and therefore either the second or the third outcome of (2). Using symmetry, we deduce that a similar statement holds for M. This proves (2).

(3) $\{p_2, \dots, p_{k-1}\}$ is anticomplete to $V(H)$.

Suppose for some $2 \leq i \leq k-1$ p_i has a neighbour y in $V(H)$. Assume first that y is non-adjacent to w, and let C be the w-gap of H containing y. Let x and z be the borders of C. If p_1 has a neighbour $n \in V(H) \setminus (V(C) \cup \{x, z\})$, then the path from n to y with interior in $\{p_1, \dots, p_i\}$ contradicts the minimality of k. From the symmetry, this implies that $M \cup N \subseteq V(C) \cup \{x, z\}$, a contradiction. This proves that y is adjacent to w, and, since y is an arbitrary neighbour of p_i in $V(H)$, we deduce that $N(p_i) \cap V(H) \subseteq N(w)$.

Next assume that there exists $n \in N \setminus N(w)$. Let C be the w-gap of H containing n, and let x, z be the borders of C. By (2) and the definition of P, $M \cap N \subseteq \{x, z\}$. By the minimality of k, the path from n to y with interior in $\{p_1, \dots, p_i\}$ fails to satisfy one of the hypotheses of the theorem, and therefore $N(p_i) \cap V(H) \subseteq \{x, z\}$. Since $G|(V(H) \cup \{p_i\})$ is not a theta, p_i is adjacent to at most one of x, z. We may assume without loss of generality, that $x = y$ and p_i is non-adjacent to z.

We claim that $\{p_2, \dots, p_{k-1}\}$ is anticomplete to $V(H) \setminus \{x\}$. For suppose there exists $2 \leq j \leq k-1$, such that p_j has a neighbour in $V(H) \setminus \{x\}$. By the previous argument applied to p_j instead of p_i, we deduce that the only neighbour of p_j in $V(H)$ is z. This implies that there exists a path P' from x to z with interior in $\{p_2, \dots, p_{k-1}\}$. But now the paths x-C-z, x-w-z and x-P'-z form a theta, contrary to 2.1. This proves the claim.

If p_k is adjacent to z, let P' be the path from x to z with interior in $\{p_i, \dots, p_k\}$. Then, by the claim, the paths x-C-z, x-w-z and x-P'-z form a theta, contrary to 2.1. This proves that p_k is non-adjacent to z.

Let $v_1, v_2 \in M$ be such that the subpath S_1 of $V(H) \setminus \{z\}$ between x and v_1 and the subpath S_2 of $V(H) \setminus \{x\}$ between z and v_2 contain no vertex of M, other than v_1, v_2, respectively. Let T be the subpath of $H \setminus \{n\}$ between v_1 and v_2. Now the minimality of k and the fact that p_k is non-adjacent to z, applied to the path from y to v' with interior in $\{p_i, \dots, p_k\}$, for $v' \in (V(S_1) \cup V(T)) \cap M$, imply that $N(w) \cap (V(S_1) \cup V(T)) \subseteq \{x, v_2\}$.

Let C' be the path from z to p_1 with interior in $V(C)$. In view of the claim, let H' be the hole z-C'-p_1-P-p_k-v_2-S_2-z. Then $N(w) \cap V(H) = (N(w) \cap V(H')) \cup \{x\}$, and since w is major with respect to H, this implies that $G|(V(H') \cup \{w\})$ is an even wheel, contrary to 2.1. This proves that $N \subseteq N(w)$, and from the symmetry, $M \cup N \subseteq N(w)$.

Let Q and Q' be the two paths of H between u and v, where $y \in V(Q)$. From the minimality of k we deduce that $(Q^* \setminus \{y\}) \cap N(w) = \emptyset$, and therefore no vertex of $Q^* \setminus \{y\}$ has a neighbour in $\{p_2, \dots, p_{k-1}\}$. If some vertex y' in Q'^* has a neighbour in $\{p_2, \dots, p_{k-1}\}$, then, similarly, y' is the only neighbour of w in Q'^*, and so w has exactly four neighbours in $V(H)$, a contradiction. So no such y' exists. This proves that y is the only vertex of $V(H)$ with a neighbour in $\{p_2, \ldots, p_{k-1}\}$, and $(Q^* \setminus \{y\}) \cap N(w) = \emptyset$.

Assume that $N = \{u\}$ and $M = \{v\}$, and let H' be the hole u-Q'-v-P-u. From the minimality of k, $y \neq u, v$. But now $V(H) \cap N(w) = (V(H') \cap N(w)) \cup \{y\}$, and therefore $G|(V(H') \cup \{w\})$ is an even wheel or a theta, contrary to 2.1.

Now by (2) and the symmetry we may assume that $|M| = 2$, say $M = \{t, t'\}$, and t is adjacent to t'. If $y \notin M$, then either the subpath of $H \setminus \{t'\}$ from y to t, or the subpath of $H \setminus \{t\}$ from y to t', contains a neighbour of w in its interior. From the symmetry we may assume the former. But now the path from y to t with interior in $\{p_i, \ldots, p_k\}$ contradicts the minimality of k. This proves that $y \in M$, and we may assume that $y = t$. Let T be the path from p_1 to t' with interior in $V(H) \setminus \{t\}$, and let H' be the hole p_1-P-p_k-t'-T-p_1. By the symmetry, either $|N| = 1$, or $y \in N \cap M$. In either case, $V(H) \cap N(w) = (V(H') \cap N(w)) \cup \{t\}$, and therefore $G|(V(H') \cup \{w\})$ is an even wheel or a theta, contrary to 2.1. This proves (3).

(4) There do not exist two non-adjacent vertices in N.

Suppose there exist two non-adjacent vertices in N. By (2), there exists a w-gap C with borders x, y such that $N \subseteq V(C) \cup \{x, y\}$. Since N contains two non-adjacent vertices, there exist paths S_x from p_1 to x and S_y from p_1 to y such that $V(S_x) \cup V(S_y) \subseteq V(C) \cup \{x, y, p_1\}$ and $V(S_x) \setminus \{p_1\}$ is anticomplete to $V(S_y) \setminus p_1$. Let x', y' be the neighbours of x and y, respectively, in $V(H) \setminus V(C)$. Let T be the subpath of $H \setminus \{x\}$ from x' to y'. Since w is major, w has a neighbour in T^*. Assume that p_k also has a neighbour in T^*. Then there exists a path S_z from p_k to w with interior in T^*. But now, by (3), the paths p_1-S_x-x-w, p_1-S_y-y-w and p_1-P-p_k-S_z-z-w form a theta, contrary to 2.1. This proves that p_k has no neighbour in T^*. It follows from the definition of P that p_k is adjacent to at least one of x', y'; and from the minimality of k, that p_k is adjacent to exactly one of them, say x' and not y'. If p_k is adjacent to y, let R be the path y'-y-p_k, and if p_k is non-adjacent to y, let R be the path y'-y-S_y-p_1-P-p_k. Then, by (3), y'-R-p_k-x'-T-y' is a hole, say H', and $V(H) \cap N(w) = V(H') \cap N(w) \cup \{x\}$. Since w is major with respect to H, it follows that $G|(V(H') \cup \{w\})$ is an even wheel or a theta, contrary to 2.1. This proves (4).

By (4) either $|N| = 1$ or $|N| = 2$ and the two members of N are adjacent, and from the symmetry the same holds for M. If $|N| = |M|$ then $G|(V(H) \cup V(P))$ is a theta, a prism, or an even wheel, contrary to 2.1, so we may assume that $|N| = 1$ and $|M| = 2$, and the two members of M are adjacent. But now the theorem follows by 4.8. This completes the proof of 4.9. ∎

Let H be a hole, and let w be a major vertex with respect to H. Let P be a path with vertices p_1-$\ldots$-p_k. We say that P is (H, w)-*significant* (or just *significant* when there is no risk of confusion) if (possibly with p_1 and p_k exchanged) p_1 has a neighbour $u \in V(H)$ that belongs to some w-gap, say C, of H, and p_k has a neighbour $v \in V(H)$, non-adjacent to u, and either

1. v belongs to a w-gap of H different from C, or
2. v is adjacent to w, and each of the two paths of H between u and v contains an even number of neighbours of w.

4.10 *Let G be an even-hole-free graph, and let H be a hole in G and w a major vertex with respect to H. Then every (H, w)-significant path contains a neighbour of w.*

Suppose not. Let the vertices of H be h_1-…-h_k-h_1. Let P be an (H,w)-significant path with vertices p_1-…-p_m such that w is anticomplete to $V(P)$, and assume that subject to that m is minimum. Let u and v be as in the definition of (H,w)-significant path. By 4.9 there exist $a,b \in \{1,\ldots,m\}$ such that the subpath P' of P between p_a and p_b is an (H,w)-pyramid path. We may assume that h_1 is the apex of P', and for some $i \in \{2,\ldots,k\}$, h_ih_{i+1} is the base. Let S be the path from h_1 to p_1 with interior in $\{p_2,\ldots,p_a\}$. Since $G|(V(H)\cup\{w\})$ is not an even wheel or a theta by 2.1, it follows that w has an odd number of neighbours in $V(H)$.

(1) Let $s,t \in \{2,\ldots,k\}$ with $s < t$ be such that w is adjacent to h_s and not to h_t. Let H_1 be the vertex set of the subpath of $H\setminus\{h_1\}$ between h_s and h_t, and let $H_2 = (V(H)\setminus V(H_1))\cup\{h_s,h_t\}$. Then H_1 contains an even number of neighbours of w if and only if H_2 does.

For suppose the parity of $N(w)\cap H_1$ is different from that of $N(w)\cap H_2$. Then, since h_s is adjacent to w and h_t is not, $V(H)$ contains an even number of neighbours of w, and at least four since w is major, contrary to 2.1. This proves (1).

(2) No vertex of P^* has a neighbour in $V(H)\setminus N(w)$, and no vertex of P^* has two adjacent neighbours in $V(H)$. In particular, $b \in \{1,m\}$.

From the symmetry we may assume that p_1 is adjacent to u. If some vertex $p \in P^*$ has a neighbour in $V(H)\setminus N(w)$, then one of the two paths p_1-P-p and p-P-p_m is significant, contrary to the minimality of m. Now suppose that some vertex p of P^* has two neighbours h_j,h_{j+1} in $V(H)$. Then both h_j and h_{j+1} are adjacent to w, and there exists a path of H from u to $\{h_j,h_{j+1}\}$ containing an even number of neighbours of w. But now, by (1), p_1-P-p is significant, contrary to the minimality of m. Since p_b has two adjacent neighbours in $V(H)$, we deduce that $b \in \{1,m\}$. This proves (2).

By (2) and from the symmetry we may assume that $b = m$. Since P' is an (H,w)-pyramid path, it is not significant, and therefore $a \neq 1$. Let Q be the subpath of $H\setminus\{h_{i+1}\}$ from h_1 to h_i and let Q' be the subpath of $H\setminus\{h_i\}$ from h_1 to h_{i+1}. Let $s \in \{2,\ldots,k\}$ be minimum and $t \in \{2,\ldots,k\}$ maximum such that w is adjacent to h_s and h_t.

(3) $\{p_2,\ldots,p_a\}$ is anticomplete to $V(H)\setminus\{h_1,h_s,h_t\}$.

Suppose not and let n be maximum such that p_n has a neighbour $h \in V(H)\setminus\{h_1,h_s,h_t\}$. By (2) h is adjacent to w, and by 4.9 applied to the path p_n-P-p_a, this path contains a pyramid subpath P'', and therefore some vertex p of P'' has two neighbours h_j and h_{j+1} in $V(H)$, contrary to (2). This proves (3).

(4) $\{p_2,\ldots,p_a\}$ is anticomplete to $V(H)\setminus\{h_1\}$.

By (3), it is enough to show that $\{p_2,\ldots,p_a\}$ is anticomplete to $\{h_s,h_t\}$. Suppose not, and let n be maximum such that p_n is adjacent to one of h_s,h_t. Since h_s-p_n-h_t-w-h_s is not a hole of length four, p_n is adjacent to exactly one of h_s,h_t, and from the symmetry we may assume that p_n is adjacent to h_s. Assume first that $s > 2$, and let P'' be the path from h_1 to h_s with interior in

$\{p_n, p_{n+1}, \dots, p_a\}$. Then the paths h_1-h_2-H-h_s, h_1-P''-h_s and h_1-w-h_s form a theta, contrary to 2.1. So $s = 2$. Since w is major, w has a neighbour in $V(H) \setminus \{h_k, h_1, h_2, h_3\}$. Since both h_1 and h_2 are adjacent to w, and P' is an (H, w)-pyramid path, it follows that $i > 2$, and therefore p_m also has a neighbour in $V(H) \setminus \{h_k, h_1, h_2, h_3\}$. Let T be a path between w and p_m with interior in $V(H) \setminus \{h_k, h_1, h_2, h_3\}$, let H_1 be the hole h_2-Q-h_i-p_m-P-p_n-h_2, and let H_2 be the hole h_2-p_n-Pp_m-T-w-h_2. Then $N(h_1) \cap V(H_2) = (N(h_1) \cap V(H_1)) \cup \{w\}$, and since h_1 has at least two neighbours on $V(H_1)$, namely p_a and h_2, it follows that one of $G|(V(H_1) \cup \{h_1\})$ and $G|(V(H_2) \cup \{h_1\})$ is an even wheel or a theta, contrary to 2.1. This proves (3).

(5) p_1 is anticomplete to one of $V(Q) \setminus \{h_1\}$ and $V(Q') \setminus \{h_1\}$.

Assume that p_1 has a neighbour in $V(Q) \setminus \{h_1\}$ and a neighbour in $V(Q') \setminus \{h_1\}$. Let $q > 1$ and $r \leq k$ be minimum and maximum such that p_1 is adjacent to h_q and h_r. Since P is significant, the subpath of $H \setminus \{h_1\}$ between h_q and h_r contains a neighbour of w in its interior. Since h_1 is adjacent to w, by 4.9, p_1 forms an (H, w)-pyramid path. By the minimality of m, some neighbour of p_m is anticomplete to $V(P) \setminus \{p_m\}$, and in particular p_1 is non-adjacent to at least one of h_i, h_{i+1}. From the symmetry we may assume that p_1 has two neighbours h_n, h_{n+1} in $V(Q)$ and one neighbour h_r in $V(Q') \setminus \{h_1\}$. Since the paths p_1-h_r-Q'-h_1, p_1-S-h_1 and p_1-h_n-Q-h_1 do not form a theta by 2.1, it follows that p_1 is adjacent to h_1, and therefore $n = 1$. Since h_1-w-h_r-p_1-h_1 is not a hole of length four, we deduce that $r = k$. Now, $G|(V(H) \setminus \{h_1\} \cup \{p_1\})$ is a hole, say H', and $N(w) \cap V(H) = (N(w) \cap V(H')) \cup \{h_1\}$. Since w is major with respect to H, this implies that $G|(V(H') \cup \{w\})$ is an even wheel or a theta, contrary to 2.1. This proves (5).

By (5) and from the symmetry we may assume that p_1 is anticomplete to $V(Q') \setminus \{h_1\}$.

(6) Not both p_1 and w have neighbours in $Q^* \setminus \{h_2\}$.

Assume for a contradiction that both p_1 and w have a neighbour in $Q^* \setminus \{h_2\}$. Then there exists a path R from p_1 to h_i with $R^* \subseteq Q^* \setminus \{h_2\}$ and a path T from p_1 to w with $T^* \subseteq Q^* \setminus \{h_2\}$. Let h_j be the neighbour of w in T. Assume first that T can be chosen so that for some t', such that $j < t' < k$, w has a neighbour $h_{t'}$ in $V(H)$, anticomplete to $V(T) \setminus \{w\}$. Let T' be a path from p_m to $h_{t'}$ with interior in $V(H) \setminus (V(T) \cup \{h_1, h_2, \dots, h_j\} \cup \{h_k\})$. Let H_1 be the hole p_a-P-p_1-R-h_i-p_m-P'-p_a and let H_2 be the hole p_a-P-p_1-T-w-$h_{t'}$-T'-p_m-P'-p_a. Then $N(h_1) \cap V(H_2) = N(h_1) \cap V(H_1) \cup \{w\}$. Since $G|(V(H_i) \cup \{h_1\})$ is not an even wheel or a theta for $i = 1, 2$ by 2.1, we deduce that h_1 is adjacent to p_{a-1} and anticomplete to $V(P) \setminus \{p_{a-1}, p_a\}$. But now the three paths p_{a-1}-P-p_1-R-h_i, p_a-P'-p_m and h_1-Q'-h_{i+1} form a prism, contrary to 2.1. This proves that we cannot choose such T and t'. Since P' is an (H, w)-pyramid path, and so w has an odd number of neighbours in $V(Q')$, we deduce that w is anticomplete to $V(Q') \setminus \{h_1\}$, and consequently $t \leq i$.

Next we observe that if p_1 has a neighbour h_j with $j > t$, then $t < i$, and p_1 has a neighbour in the w-gap containing h_i and h_{i+1}, and therefore p_1 is a significant path, contrary to the minimality of m.

Since w is major, w has a neighbour in $Q^* \setminus \{h_2, h_{t-1}, h_t\}$, and since we cannot choose T with $T^* \subseteq Q^* \setminus \{h_2, h_{t-1}, h_t\}$ and $t' = t$, it follows that $N(p_1) \cap Q \subseteq \{h_1, h_2, h_{t-1}, h_t\}$. Since the paths h_1-S-p_1-h_t, h_1-w-h_t and h_1-Q'-h_{i+1}-h_i-Q-h_t do not form a theta by 2.1, it follows that p_1 is non-adjacent to h_t. Since p_1 has a neighbour in $Q^* \setminus \{h_2\}$, it follows that p_1 is adjacent to h_{t-1}. If w is non-adjacent to h_{t-1}, then the paths h_1-S-p_1-h_{t-1}-h_t, h_1-w-h_t and h_1-Q'-h_{i+1}-h_i-Q-h_t from a theta, contrary to 2.1, so h_{t-1} is adjacent to w. Since w is major, it follows that $t > 4$; and since h_1-p_1-h_{t-1}-w-h_1 is not a hole of length four, p_1 is non-adjacent to h_1. Since $G|(V(H) \cup \{p_1\})$ is not a theta by 2.1, it follows that p_1 is non-adjacent to h_2, and therefore h_{t-1} is the unique neighbour of p_1 in $V(H)$. Let H_3 be the hole h_1-Q-h_{t-1}-p_1-S-h_1. Then $V(H) \cap N(w) = (V(H_3) \cap V(w)) \cup \{h_t\}$, and since w is major with respect to H, it follows that $G|(V(H_3) \cup \{w\}$ is an even wheel or a theta, contrary to 2.1. This proves (6).

(7) p_1 has a neighbour in $Q^* \setminus \{h_2\}$.

Suppose not, and so $N(p_1) \cap V(H) \subseteq \{h_1, h_2, h_i\}$. Since P is significant, p_1 is adjacent to h_2. Since P is significant and P' is a pyramid, w has an even number of neighbours, and at least two, in $V(Q) \setminus \{h_1\}$. Let R be a path from h_2 to h_i with interior in $V(P)$, and let H' be the hole h_2-Q-h_i-R-h_2. Since by 2.1 $G|(V(H') \cup \{w\})$ is not an even wheel or a theta, it follows that w has exactly two neighbours in $V(Q) \setminus \{h_1\}$, and they are adjacent, say h_n and h_{n+1}. Since w is major with respect to H, we deduce that w has a neighbour in $V(Q') \setminus \{h_1\}$. Let $t' \in \{i+1, \dots, k\}$ be minimum such that w is adjacent to $h_{t'}$.

Assume first that p_1 is non-adjacent to h_i. Then $p_m \in V(R)$, and the paths h_{n+1}-Q-h_i, h_n-Q-h_2-R-p_m and w-$h_{t'}$-Q'-h_{i+1} form a prism if $i = n+1$, or an even wheel if $i \neq n+1$, contrary to 2.1. This proves that p_1 is adjacent to h_i.

Suppose p_1 is adjacent to h_1. Since h_1-p_1-h_i-w-h_1 is not a hole of length four, it follows that h_i is non-adjacent to w. We deduce from the minimality of m that h_2 is adjacent to w. But now, by (1), each of the subpaths of H between h_1 and h_i contains an even number of neighbours of w, and therefore p_1 is a significant path, contrary to the minimality of m. This proves that p_1 is non-adjacent to h_1.

Since $G|(V(H) \cup \{p_1\})$ is not a theta by 2.1, $i = 3$, and so w is adjacent to both h_2 and h_3. But now by 4.9 the path p_1-P-p_a contains a subpath that is an (H, w)-pyramid path, which is impossible since h_1, h_2, h_3 are the only vertices of $V(H)$ with a neighbour in p_1-P-p_a and all of them are adjacent to w. This proves (7).

By (6) and (7) w has no neighbour in $Q^* \setminus \{h_2\}$. Since P' is an (H, w)-pyramid path, w is adjacent to both or neither of h_2 and h_i. Since P is a significant path, we deduce that w is adjacent to both h_2 and h_i. Let T be the path from p_1 to h_i with interior in $Q^* \setminus \{h_2\}$, and let H' be the hole h_1-S-p_1-T-h_i-h_{i+1}-Q'-h_1. Then $N(w) \cap V(H) = (N(w) \cap V(H')) \cup \{h_2\}$, and $G|(V(H') \cup \{w\})$ is an even wheel or a theta, contrary to 2.1. This completes the proof of 4.10. ∎

We can now prove 4.5.

Proof of 4.5. It is enough to prove that w has a neighbour in the interior of every path of G from $V(C)$ to $A \cup B$. Let P be such a path. Then P is (H, w)-significant, and so by 4.10 w has a neighbour in P^*. This proves 4.5. ∎

5 Non-dominating cliques

In view of 3.1, to complete the proof of 1.2, it is enough to prove the following:

5.1 *Let G be an even-hole-free graph such that 1.2 is true for all graphs with fewer vertices than G. Let K be a non-dominating clique of G of size at most two. Then there is a vertex in $V(G) \setminus N(K)$ which is bisimplicial in G.*

Proof. Assume no such vertex exists. By 2.4 we may assume that $K = \{x, y\}$ with $x, y \in V(G)$. Let $C = N(x) \cap N(y) \setminus \{x, y\}$. Then the five sets $N(x) \setminus N(y), N(y) \setminus N(x), V(G) \setminus N(K), C$ and $\{x, y\}$ partition $V(G)$.

(1) $V(G) \setminus N(K)$ is connected, every vertex of $N(K) \setminus K$ has a neighbour in $V(G) \setminus N(K)$, and there exists a hole H of G with $\{x, y\} \subseteq V(H)$.

If $N(x) \setminus N(y)$ is empty, let G' be the graph $G \setminus \{x\}$. Then $\{y\}$ is a non-dominating clique in G'. By the minimality of $|V(G)|$, there is a vertex $v \in V(G') \setminus N(y)$ that is bisimplicial in G', and hence in G. Since $N(x) \setminus N(y)$ is empty, v is non-adjacent to x, and therefore v is a bisimplicial vertex of G, and $v \in V(G) \setminus N(K)$, a contradiction. This proves that $N(x) \setminus N(y)$, and from the symmetry $N(y) \setminus N(x)$, are non-empty. Consequently, the third assertion of (1) follows from the first two.

If some vertex $v \in N(K) \setminus K$ is anticomplete to $V(G) \setminus N(K)$, let $X = \{v\}$, and otherwise, if $V(G) \setminus N(K)$ is not connected, let X be a component of $V(G) \setminus N(K)$. In either case $V(G) \setminus (X \cup N(K))$ is non-empty, and therefore K is a non-dominating clique in $G' = G \setminus X$. Since $V(G') \setminus N(K) \subseteq V(G) \setminus N(K)$, the minimality of $|V(G)|$ implies that there exists a vertex $w \in V(G) \setminus N(K)$ that is bisimplicial in G'. But it follows from the definition of X that w is anticomplete to X, and so $N_G(w) = N_{G'}(w)$ and w is bisimplicial in G, a contradiction. This proves (1).

(2) Let H be a hole with $x, y \in V(H)$. Then H is dominating. If $w \in C$ is major with respect to H, then w is complete to $V(H)$.

Suppose first that H is non-dominating. Then by 3.1 there is a vertex $v \in V(G) \setminus N(H)$ that is bisimplicial in G, and therefore $v \in V(G) \setminus N(K)$, a contradiction. This proves that H is dominating.

Let $w \in C$ be a major vertex with respect to H, and suppose that w is not complete to $V(H)$. Now by 4.5 there exists a subset N' of $N(w)$ such that $N' \cup \{w\}$ is a star cutset in G, such that $\{x, y\} \not\subseteq N'$ and some component of $G \setminus (N' \cup \{w\})$ is disjoint from $\{x, y\}$ and not complete to w, contrary to 4.1. This proves (2).

(3) No vertex of C has both a neighbour in $N(x) \setminus N(y)$ and a neighbour in $N(y) \setminus N(x)$.

Let $A = N(x) \setminus N(y)$, $B = N(y) \setminus N(x)$ and $D = V(G) \setminus N(K)$. Suppose some $c \in C$ has a neighbour in A and a neighbour in B. Let A' be the set of neighbours of c in A, $A'' = A \setminus A'$, and let B', B'', D', D'' be defined similarly.

We claim that A'' is non-empty. For suppose not. Assume first that $D'' \neq \emptyset$. Then $\{y, c\}$ is a non-dominating clique in $G' = G \setminus \{x\}$. By the minimality of $|V(G)|$, there exists a vertex v in

$V(G') \setminus N(\{y,c\})$ that is bisimplicial in G', and since $A \cup B \cup C \subseteq N(\{y,c\})$, it follows that v is in D. But $N_G(v) = N_{G'}(v)$, because x is anticomplete to D, and consequently v is bisimplicial in G, a contradiction. This proves that $D'' = \emptyset$. Applying 1.2 to the graph $G'' = G \setminus \{c\}$ and using the minimality of $|V(G)|$, we deduce that there exists a vertex v of $V(G'') \setminus N(K)$ that is bisimplicial in G''. Since $A \cup B \cup C \subseteq N(K)$ and $D = D'$, it follows that v is in D and $N_G(v) = N_{G''}(v) \cup \{c\}$. Since y-c-v-b-y is not a hole of length four for any $b \in N(v) \cap (B \cup C)$, it follows that $N_{G''}(v) \cap (B \cup C)$ is complete to c. From the symmetry, $N_{G''}(v) \cap (A \cup C)$ is complete to c; and therefore, since c is complete to D, it follows that $N_{G''}(v)$ is complete to c. Consequently, v is bisimplicial in G, a contradiction. This proves the claim.

From the symmetry it follows that both A'' and B'' are non-empty. Since a-x-y-b-a is not a hole of length four for $a \in A$ and $b \in B$, it follows that A is anticomplete to B. Choose $a' \in A', b' \in B', a'' \in A''$ and $b'' \in B''$. By (1), there exists a path P_1 from a' to b'' and a path P_2 from a'' to b', both with interior in D. Let H_1, H_2 be the holes x-a'-P_1-b''-y-x and x-a''-P_2-b'-y-x. By (2), and since c is non-adjacent to a'' and b'', it follows that c is not major with respect to H_1 or H_2, and therefore c is anticomplete to $P_1^* \cup P_2^*$.

We claim that $V(P_1)$ is disjoint from $V(P_2)$, and $V(P_1) \setminus \{b''\}$ is anticomplete to $V(P_2) \setminus \{a''\}$. Suppose not. Then there is a path P from a' to b' with $P^* \subseteq P_1^* \cup P_2^*$, and the hole x-a'-P-b'-y-x contains exactly four neighbours of w, contrary to 2.1. This proves the claim.

Let d_1 be the neighbour of b'' in P_1 and d_2 the neighbour of a'' in P_2. Since A is anticomplete to B, it follows that d_1 and d_2 are in D. By (2), d_1 has a neighbour in $V(H_2)$, and by the argument of the previous paragraph, d_1 is adjacent to a'' and not to d_2. Similarly, d_2 is adjacent to b''. But now, since A is anticomplete to B, a''-d_2-b''-d_1-a'' is a hole of length four, a contradiction. This proves (3).

Let m be the minimum length of all holes containing x and y.

(4) Let H be a hole with $\{x,y\} \subseteq V(H)$. If a vertex w of G is major with respect to H, then w is not complete to $\{x,y\}$. Moreover, if H has length m, then no vertex of G is major with respect to H, and every pyramid with respect to H is adjacent to both x and y.

If w is a major vertex with respect to H that is complete to $\{x,y\}$, then by (3) w is not complete to $V(H)$, contrary to (2).

If H has length m and w is a major vertex or a pyramid with respect to H, then the minimality of $|V(H)|$ implies that w is adjacent to both x and y, and the result follows. This proves (4).

Let

$$W = \bigcup \{V(H) \ : \ H \text{ is a hole, } K \subseteq V(H) \text{ and } |V(H)| = m\}.$$

For $1 \le i \le m-2$ let A_i be the set of all vertices $v \in W$ such that there exists a hole H of length m with $x, y, v \in V(H)$, and the subpath of $H \setminus \{y\}$ from x to v has length i. Let $A_0 = \{x\}$ and $A_{m-1} = \{y\}$. Clearly $W = \bigcup_{i=0}^{m-1} A_i$.

(5) $A_i \cap A_j = \emptyset$ and A_i is anticomplete to A_j for all $i, j \in \{0, \dots, m-1\}$ with $1 < j - i < m - 1$.

Since for $1 \le i \le m-2$ every vertex in A_i has a neighbour in A_{i-1} and in A_{i+1}, it is enough to prove the second statement. Suppose for some $1 \le i < j \le m-2$ with $j-i>1$ there exist $a_i \in A_i$ and $a_j \in A_j$ that are adjacent. By the definition of A_i and A_j, there exists a path P from a_i to x, such that $y \notin V(P)$ and P has length i, and a path Q from a_j to y, such that $x \notin V(Q)$ and Q has length $m-j-1$. Since every vertex of W is in a hole containing x and y, it follows that no vertex of $V(P) \cup V(Q)$ is adjacent to both x and y, and therefore $G|(V(P) \cup V(Q))$ contains a hole H' with $x, y \in V(H')$. But $|V(P) \cup V(Q)| < m$, contrary to the minimality of m. This proves (5).

Let $1 \le i \le m-2$ and let $u \in A_i$. We say that a path P is an *x-path* for u if

- $u \in V(P)$.
- for $0 \le j \le i$ $|V(P) \cap A_j| = 1$, and
- for $V(P) \subseteq \bigcup_{j=0}^{i} A_j$.

and P is a *y-path* for u if

- $u \in V(P)$.
- for $i \le j \le m-1$, $|V(P) \cap A_j| = 1$.
- $V(P) \subseteq \bigcup_{j=i}^{m-1} A_j$.

By the definition of W, there is an x-path and a y-path for every vertex in $W \setminus K$. It follows from (5) that for every $u \in W \setminus K$, if P is an x-path for u and Q is a y-path for u, then $V(P) \cap V(Q) = \{u\}$, $V(P) \setminus \{u, x\}$ is anticomplete to $V(Q) \setminus \{u, y\}$, and $G|(V(P) \cup V(Q))$ is a hole of length m. Moreover, let $u \in A_i$ and let u' be a neighbour of u in A_{i-1}. Then there is an x-path P' for u', and the path x-P'-u'-u is an x-path for u. Thus every neighbour of u in A_{i-1} is in an x-path for u, and similarly every neighbour of u in A_{i+1} is in a y-path for u.

Let us call a pair of non-adjacent vertices u, v in A_i an *x-pair* if $N(u) \cap N(v) \cap A_{i-1} \neq \emptyset$, and a *$y$-pair* if $N(u) \cap N(v) \cap A_{i+1} \neq \emptyset$,

(6) Let $1 \le i \le m-2$ and let $u, v \in A_i$ be non-adjacent. Then u, v is either an x-pair, or a y-pair, and not both. Moreover,

- *if u, v is an x-pair, then $N(u) \cap A_{i-1} = N(v) \cap A_{i-1}$, $N(u) \cap N(v) \cap A_{i+1} = \emptyset$, and $N(u) \cap A_{i+1}$ is complete to $N(v) \cap A_{i+1}$*
- *if u, v is a y-pair, then $N(u) \cap A_{i+1} = N(v) \cap A_{i+1}$, $N(u) \cap N(v) \cap A_{i-1} = \emptyset$, and $N(u) \cap A_{i-1}$ is complete to $N(v) \cap A_{i-1}$.*

Let P_u and Q_u be an x-path and a y-path for u, respectively; and let P_v and Q_v be defined similarly. Since x-P_u-u-Q_u-y-x is a hole, by (2) and (5) v has a neighbour in $(V(P_u) \cup V(Q_u)) \cap (A_{i-1} \cup A_i \cup A_{i+1})$. Since v is non-adjacent to u, we may assume from the symmetry that v is adjacent to the neighbour p of u in P_u, and hence u, v is an x-pair. So $p \in A_{i-1}$. Since by (5) A_{i-1} is anticomplete to A_{i+1}, and u-p-v-a-u is not a hole of length four for any $a \in A_{i+1}$, it follows that $N(u) \cap N(v) \cap A_{i+1} = \emptyset$. Therefore u, v is not a y-pair.

Suppose there exist $a \in N(u) \cap A_{i+1}$ and $a' \in N(v) \cap A_{i+1}$ such that a is non-adjacent to a'. Then $i < m-2$. Since every vertex in $N(u) \cap A_{i+1}$ is in a y-path for u, we may assume that $a \in Q_u$. By (2), a' has a neighbour in $V(P_u) \cup V(Q_u)$. By (5) and since a' is anticomplete to $\{u, a\}$, a' is adjacent to the unique vertex q of $V(Q_u) \cap A_{i+2}$. But now, again by (5), p-v-a'-q-a-u-p is a hole of length six, a contradiction. This proves that $N(u) \cap A_{i+1}$ is complete to $N(v) \cap A_{i+1}$.

It remains to prove that $N(u) \cap A_{i-1} = N(v) \cap A_{i-1}$. Suppose there exists $p' \in (N(v) \cap A_{i-1}) \setminus N(u)$. Since every vertex in $N(v) \cap A_{i-1}$ is in an x-path for v, we may assume that $p' \in V(P_v)$. Since x-P_v-v-Q_v-y-x is a hole, and by (2), and (5), u has a neighbour in $(V(P_v) \cup V(Q_v)) \cap (A_{i-1} \cup A_i \cup A_{i+1})$. But u is anticomplete to $\{p', v\}$ and $N(u) \cap N(v) \cap A_{i+1} = \emptyset$, a contradiction. This proves (6).

(7) Let $1 \leq i \leq m-2$ and let $u, v \in A_i$ be an x-pair. Then A_i is complete to $N(u) \cap A_{i-1}$. In particular there is no y-pair in A_i.

Suppose there exists $z \in N(u) \cap A_{i-1}$ and $w \in A_i$ such that z is non-adjacent to w. Since u, v is an x-pair, v is adjacent to z, and there exist $a, b \in A_{i+1}$ such that u is adjacent to a and not to b, v is adjacent to b and not to a, and a is adjacent to b. Since z-u-w-v-z is not a hole of length four, we may assume from the symmetry that w is non-adjacent to v. Since z is in an x-path for v, and b is in a y-path for v, it follows from (2) and (5) that w is adjacent to b. Now by (6), v, w is a y-pair, and therefore there exists $z' \in A_{i-1}$, adjacent to z and w and non-adjacent to v. Since u, v is an x-pair, u is non-adjacent to z'. Since z' is in an x-path for w, and b is in a y-path for w, it follows from (2) and (5) that u is adjacent to w. But now z-u-w-z'-z is a hole of length four, a contradiction. This proves (7).

(8) Let $u \in V(G)$ be adjacent to x or to y, or such that K is non-dominating in $G \setminus \{u\}$. Then there exists a neighbour v of u, such that v is anticomplete to $\{x, y\}$ and v is bisimplicial in the graph $G \setminus \{u\}$.

First we claim that K is non-dominating in $G \setminus \{u\}$. To prove the claim, we may assume that $u \in N(K)$. But in this case, since K is non-dominating in G, there exists a vertex $v \neq u$, such that v is anticomplete to K, and therefore K is non-dominating in $G \setminus \{u\}$. This proves the claim.

We deduce from the minimality of $|V(G)|$ that there exists a vertex v in $V(G) \setminus (N(K) \cup \{u\})$ that is bisimplicial in $G \setminus \{u\}$. But now, since v is not bisimplicial in G, it follows that u is adjacent to v. This proves (8).

(9) Let $1 \leq j < k \leq m-2$ such that $k - j > 1$ and let $a_j \in A_j$ and $a_k \in A_k$. Then there exists a path from a_j to a_k with interior in $\bigcup_{i=j+1}^{k-1} A_i$ using exactly one vertex from each of $A_j, A_{j+1}, \ldots A_k$.

Let Q be a y-path for a_j and let a_{k-1} be the vertex of Q in A_{k-1}. We may assume that a_k is non-adjacent to a_{k-1}, for otherwise by (5) a_j-Q-a_{k-1}-a_k is a path and the claim holds. Let a'_{k-1} be a neighbour of a_k in A_{k-1} and let a_{k-2} be the vertex of Q in A_{k-2}. Then we may assume that a'_{k-1} is non-adjacent to a_{k-2}, for otherwise by (5) a_j-Q-a_{k-2}-a'_{k-1}-a_k is a path and the claim holds. If a_{k-1} is adjacent to a'_{k-1} then x-P_j-a_j-Q-a_{k-1}-a'_{k-1}-a_k-Q_k-y-x is a hole of length $m+1$, and therefore even, where P_j is an x-path for a_j and Q_k is a y-path for a_k, a contradiction. So a_{k-1} is not adjacent to a'_{k-1}. But now, since a_{k-1} is non-adjacent to a_k, and a'_{k-1} is non-adjacent to a_{k-2}, the pair

a_{k-1}, a'_{k-1} is not an x-pair and not a y-pair, contrary to (6). This proves (9).

(10) Let v be a vertex adjacent to both x and y. Then there exists an odd integer $1 \leq i \leq m-2$ such that $N(v) \cap W \subseteq K \cup A_i$.

Let $j > 0$ be minimum and $k < m-1$ maximum such that v has a neighbour $a_j \in A_j$ and $a_k \in A_k$. Let P_j be an x-path for a_j and let Q_k be a y-path for a_k. Since x-P_j-a_j-v-x and y-Q_k-a_k-v-y are not even holes, it follows that $k-j$ is even, and in particular either $k = j$ or $k - j > 1$. Suppose $j \neq k$ and let R be a path from a_j to a_k as in (9). Then x-P_j-a_j-R-a_k-Q_k-y-x is a hole of length m and v is a major vertex with respect to it, contrary to (4). This proves that $k = j$. But now, since x-P_j-a_j-v-x is not an even hole, P_j is odd, and therefore j is odd. This proves (10).

(11) Let $v \in V(G) \setminus W$ and let $N = N(v) \cap W$. Then either

1. *$N = A_i$ for some $1 \leq i \leq m-2$, or*
2. *For some $1 \leq i \leq m-3$, $N \subseteq A_i \cup A_{i+1}$, $N \cap A_i \neq \emptyset$, $N \cap A_{i+1} \neq \emptyset$, $N \cap A_i$ is complete to $N \cap A_{i+1}$, and $A_i \setminus N$ is anticomplete to $A_{i+1} \setminus N$, or*
3. *$K \subseteq N \subseteq A_i \cup K$ for some odd $1 \leq i \leq m-2$, and $N \cap A_i \neq \emptyset$, or*
4. *$N = K$, or*
5. *$x \in N \subseteq A_1 \cup \{x\}$, and $N \cap A_1 \neq \emptyset$, or*
6. *$y \in N \subseteq A_{m-2} \cup \{y\}$, and $N \cap A_{m-2} \neq \emptyset$.*

If $K \subseteq N$, then by (10) either the third or the fourth outcome holds, so, from the symmetry we may assume that $y \notin N$. Assume that $x \in N$. Then from the minimality of m we deduce that v is anticomplete to A_i for $i > 2$, and since $v \notin W$, it follows that v is anticomplete to A_2. If v has a neighbour in A_1, then the fifth outcome holds, so we may assume not. By (8), v has a neighbour $u \in V(G) \setminus N(K)$, and we have just shown that $u \notin W$. By (2) u has a neighbour in W; let j be maximum such that u has a neighbour in A_j and let a_j be such a neighbour. Then $j < m-1$. Since x-v-u-a_1-x is not a hole of length four for any $a_1 \in A_1$, u is anticomplete to A_1. Let Q be a y-path for a_j. Then x-v-u-a_j-Q-y-x is a hole of length at most $m+1$, and since G is even-hole-free, it is a hole of length m. But now $v \in A_1$, a contradiction. Thus we may assume that $N \cap K = \emptyset$.

Let $1 \leq j \leq m-2$ be minimum and $1 \leq k \leq m-2$ maximum such that v has a neighbour $a_j \in A_j$ and $a_k \in A_k$. Let P_j be an x-path for a_j and let Q_k be a y-path for a_k. If $k - j > 1$, then x-P_j-a_j-v-a_k-Q_k-y-x is a hole of length at most m containing x and y, which contradicts either the minimality of m or the fact that $v \notin W$, so either $j = k$ or $j = k-1$. If $j = k$, then by (2) v is complete to A_j, and the first outcome holds. So we may assume that $j = k-1$. To show that the second outcome holds, it remains to prove that $N \cap A_j$ is complete to $N \cap A_k$, and $A_j \setminus N$ is anticomplete to $A_k \setminus N$. Let $u \in A_j$ and $w \in A_k$, let P be an x-path for u and let Q be a y-path for w. Assume first that $u \in N \cap A_j$, $w \in N \cap A_k$ and u is non-adjacent to w. Then x-P-u-v-w-Q-y-x is a hole of length $m+1$, and therefore even, a contradiction. This proves that $N \cap A_j$ is complete to $N \cap A_k$. Next assume that $u \in A_j \setminus N$, $w \in A_k \setminus N$ and u is adjacent to w. Then x-P-u-w-Q-y-x is a hole and v has no neighbour in it, contrary to (2). This proves that $A_j \setminus N$ is anticomplete to

$A_k \setminus N$ and completes the proof of (11).

For $1 \leq i \leq m-2$, let B_i be the set of all vertices of $V(G) \setminus W$ that are complete to A_i and have no other neighbours in W, and let C_i be the set of all vertices of $V(G) \setminus W$ that are complete to K, anticomplete to $W \setminus (K \cup A_i)$ and have at least one neighbour in A_i. For $0 \leq i \leq m-2$ let $B_{i,i+1}$ the set of all vertices of $V(G) \setminus W$ that have a neighbour in A_i and a neighbour in A_{i+1}, and are anticomplete to $W \setminus (A_i \cup A_{i+1})$. Let $B_{x,y}$ be the vertices of $V(G) \setminus W$ that are complete to K and anticomplete to $W \setminus K$. Let $B_0 = B_{m-1} = C_0 = C_{m-1} = \emptyset$. Then all these sets are pairwise disjoint, and by (11)

$$V(G) = W \cup \bigcup_{i=0}^{m-2} B_i \cup C_i \cup B_{i,i+1} \cup B_{x,y}.$$

(12) Both C_1 and C_{m-2} are cliques.

Suppose there exist two non-adjacent vertices u, u' in C_1. Since y-u-a-u'-y is not a hole of length four for any $a \in A_1$, it follows that no vertex of A_1 is adjacent to both u and u', and in particular u is not complete to A_1, and neither is u'. Let a_1 be a neighbour of u in A_1 and a_1' a neighbour of u' in A_1. By (8), u has a neighbour in $V(G) \setminus N(K)$, and since u is anticomplete to $W \setminus (K \cup A_1)$, it follows that u has a neighbour in $\bigcup_{i=1}^{m-2} B_i \cup \bigcup_{j=1}^{m-3} B_{i,i+1}$. Let n be such a neighbour.

Assume first that $n \in B_i$ for some i. Then $i > 1$, since x-u-n-a_1'-x is not a hole of length four. Let Q be a y-path for a_1 and let a_i be the vertex of Q in A_i. Since B_i is complete to A_i, n is adjacent to a_i. But now the three paths a_i-Q-a_1-u, a_i-n-u, a_i-Q-y-u form a theta, contrary to 2.1. This proves that u is anticomplete to $\bigcup_{i=1}^{m-2} B_i$.

Next assume that $n \in B_{i,i+1}$ for some $2 \leq i \leq m-3$. Let a_i, a_{i+1} be neighbours of n in A_i and A_{i+1}, respectively. By (11) a_i is adjacent to a_{i+1}. First we claim that there exist a path R from a_i to a non-neighbour of u in A_1 with interior in $\bigcup_{j=2}^{i-1} A_i$. If $i \geq 3$, the existence of such a path follows from (9), so we may assume that $i = 2$ and every neighbour a of a_i in A_1 is adjacent to u. But now u-a-a_i-n-u is a hole of length four, a contradiction. So such a path R exists, and we may assume that $R \cap A_1 = \{a_1'\}$. Let Q be a y-path for a_{i+1}. Now the three paths x-a_1'-R-a_i, u-n and y-Q-a_{i+1} form a prism, contrary to 2.1. This proves that $n \in B_{1,2}$. Similarly, u' has a neighbour n' in $B_{1,2}$.

Let a be a neighbour of n in A_1, and let a' be a neighbour of n' in A_1. Since u-x-u'-n-u is not a hole of length four, u is non-adjacent to n', and similarly u' is non-adjacent to n. Since a_1'-x-u-n-a_1' is not a hole of length four, it follows that n is non-adjacent to a_1'. Let T be a y-path for a_1'. By (11), n is adjacent to the vertex of T in A_2, say a_2. Now a is adjacent to u and, by (11), to a_2, and since a, a_1' is not an x-pair and not a y-pair, a is adjacent to a_1'. Since no vertex of A_1 is adjacent to both u and u', a is non-adjacent to u'. Since a is adjacent to a_2, a-a_2-T-y is a y-path for a. By the previous argument applied to u', n', a instead of u, n, a_1', we deduce that n' is non-adjacent to a and adjacent to a_2, and every neighbour a' of n' in A_1 is adjacent to a and u', and therefore not to u and not to n. Since n-a_2-n'-u'-y-u-n is not a hole of length six, it follows that n is adjacent to n'. But now n-n'-a'-a-n is a hole of length four, a contradiction. This proves (12).

(13) If $c_1 \in C$ has a neighbour $a \in N(x) \setminus N(y)$ and $c_2 \in C$ has a neighbour $b \in N(y) \setminus N(x)$, then c_1 and c_2 are adjacent.

Suppose not. Since a-x-y-b-a is not a hole of length four, a is non-adjacent to b. By (3) c_1 is non-adjacent to b and c_2 to a. By (1) there exists a path P from a to b such that $P^* \subseteq V(G) \setminus N(K)$. Let H_1 be the hole a-x-y-b-P-a. By (2), c_1 and c_2 are not major with respect to H_1, and therefore $\{c_1, c_2\}$ is anticomplete to P^*. Let D be a minimal connected subset of $V(G) \setminus N(K)$ such that $P^* \subseteq D$, and at least one of c_1, c_2 has a neighbour in D. Since c_1-x-c_2-d-c_1 is not a hole of length four, no vertex d of D is adjacent to both c_1 and c_2, and therefore, the minimality of D implies that exactly one of c_1, c_2 has a neighbour in D, say c_1. Let Q be a path from c_1 to b with $Q^* \subseteq D$. Now both c_1-Q-b-y-c_1 and c_1-Q-b-c_2-x-c_1 are holes and their lengths differ by one, so one of them is even, a contradiction. This proves (13).

(14) Let $c_1, c_2 \in C$ be non-adjacent. Then $\{c_1, c_2\}$ is anticomplete to $N(K) \setminus (C \cup K)$.

Suppose c_1 has a neighbour $a \in N(x) \setminus N(y)$, say. Then by (3) c_1 is anticomplete to $N(y) \setminus N(x)$, and by (13) c_2 is anticomplete to $N(y) \setminus N(x)$. Choose $b \in N(y) \setminus N(x)$. By (1) there is a path P from a to b with $P^* \subseteq V(G) \setminus N(K)$. Let H be the hole a-P-b-y-x-a. By (2), c_1 is not a major vertex with respect to H, and therefore c_1 is anticomplete to P^*. Suppose c_2 has neighbour in $V(P)$. Let Q be a path from c_2 to a with $Q^* \subseteq P$. Then both a-Q-c_2-x-a and a-Q-c_2-y-c_1-a are holes, and their lengths differ by one, so one of them is even, a contradiction. This proves that c_2 is anticomplete to $V(P)$. By (1) c_2 has a neighbour d in $V(G) \setminus N(K)$. Since c_1-x-c_2-d-c_1 is not a hole of length four, c_1 is non-adjacent to d. By (2) the hole H is dominating, and so d has a neighbour in $V(P)$. Since c_2-y-b-d-c_2 is not a hole of length four, d is non-adjacent to b, and so there exists a path R from d to a such that $R^* \subseteq V(P) \setminus \{b\}$. But now both a-x-c_2-d-R-a and a-c_1-y-c_2-d-R-a are holes, and their lengths differ by one, so one of them is even, a contradiction. This proves (14).

(15) Let $c_1, c_2 \in C$ be non-adjacent. Then every path P between c_1 and c_2 with interior in $V(G) \setminus N(K)$ has length three, the set $V(P) \cup \{x, y\}$ is dominating, and either

- *$m = 5$, $c_1, c_2 \in B_{x,y}$ and $P^* \subseteq B_2$, or*
- *$m = 7$, $c_1, c_2 \in C_3$ and $P^* \subseteq A_3$.*

Let P be a path with ends c_1, c_2 and interior in $V(G) \setminus N(K)$. Let Q be a path with ends $a \in N(x) \setminus N(y)$ and $b \in N(y) \setminus N(x)$, and with $Q^* \subseteq V(G) \setminus N(K)$. We claim that Q^* contains a vertex with a neighbour in P^*. Let H be the hole x-a-Q-b-y-x. By (2) H is dominating, and so every vertex of P^* has a neighbour in $V(H)$, and therefore in $V(Q)$. We may assume that no vertex of P^* has a neighbour in Q^*, for otherwise the claim holds, and therefore every vertex of P^* is adjacent to either a or b. Let p be the neighbour of c_1 in P. From the symmetry we may assume that p is adjacent to a. But c_1 is non-adjacent to a, by (14), and so a-x-c_1-p-a is a hole of length four, a contradiction. This proves the claim.

Next we show that every vertex of G has a neighbour in $V(P) \cup \{x, y\}$. For suppose there exists v with no such neighbour. Then v belongs to $V(G) \setminus N(K)$. Suppose there exists a path P_1 from v to $a' \in N(x) \setminus N(y)$ and a path P_2 from v to $b' \in N(y) \setminus N(x)$ with $(P_1^* \cup P_2^*) \cap N(V(P) \cup \{x, y\}) = \emptyset$. Then, in $P_1 \cup P_2$, there is a path from a' to b' that contradicts the claim of the previous paragraph. So from the symmetry we may assume that there is no path from v to $N(y) \setminus N(x)$ with interior in $V(G) \setminus N(V(P) \cup \{x, y\})$. Let F be a component of $G \setminus N(V(P) \cup \{x\})$ containing v. Then F is disjoint from $N(y) \setminus N(x)$. By 3.1 applied to $G' = G|(F \cup N(V(P) \cup \{x\}))$ and the hole x-c_1-P-c_2-x, there exists a bisimplicial vertex w of G' in

$$V(G') \setminus N_{G'}(V(P) \cup \{x\}) = V(G') \setminus N_G(V(P) \cup \{x\}) = F.$$

But now it follows from the definition of F that $N_{G'}(w) = N_G(w)$, and so, since F is disjoint from $N(K)$, w is a bisimplicial vertex of G contained in $V(G) \setminus N(K)$, a contradiction. This proves that every vertex of G has a neighbour in $V(P) \cup \{x, y\}$.

If a has a neighbour in P^* define $x_a = a$, and otherwise let x_a be the neighbour of a in Q. Let x_b be defined similarly. Then both x_a and x_b have neighbours in P^*. Let p_1 be the neighbour of x_a in P such that the subpath P_1 of P from p_1 to c_1 contains no other neighbour of x_a. Let p_2 be the neighbour of x_a in P such that the subpath P_2 of P from p_2 to c_2 contains no other neighbour of x_a. Let p_1', P_1', p_2', P_2' be defined similarly with x_b instead of x_a.

We claim that x_a (and from the symmetry x_b) has exactly two neighbours in $V(P)$ and they are adjacent to each other. Suppose first that $x_a \neq a$. Now, if p_1 and p_2 are distinct and non-adjacent, then the three paths x_a-p_1-P_1-c_1-x, x_a-p_2-P_2-c_2-x and x_a-a-x form a theta, and if $p_1 = p_2$ then then the three paths p_1-P_1-c_1-x, p_1-P_2-c_2-x and p_1-x_a-a-x form a theta, contrary to 2.1. This proves that p_1 and p_2 are distinct and adjacent, and the claim follows.

So we may assume that $x_a = a$. By (14) x_a is non-adjacent to both c_1 and c_2. Then $p_1 \neq p_2$, for otherwise, $G|(V(P) \cup \{x, x_a\})$ is a theta, contrary to 2.1; and may assume that p_1 is non-adjacent to p_2, for otherwise the claim holds. So x_a is a major vertex with respect to the hole x-c_1-P-c_2-x. But now, since x_a is non-adjacent to both c_1 and c_2, there are two disjoint x_a-gaps in this hole, and so by 4.5 G admits a full star cutset, contrary to 4.2. This proves that x_a (and from the symmetry x_b) has exactly two neighbours in $V(P)$ and they are adjacent to each other, that is, p_1 and p_2 are distinct and adjacent, and the same holds for p_1' and p_2'.

If $p_1' \in V(P_2)$, then the paths x_a-Q-a-x, p_1-P_1-c_1 and p_2-P_2-p_1'-x_b-Q-b-y form a prism or an even wheel, and if $p_2' \in V(P_1)$, then the paths x_a-Q-a-x, p_2-P_2-c_2 and p_1-P_1-p_2'-x_b-Q-b-y form a prism or an even wheel, in both cases contrary to 2.1. This proves that $p_1 = p_1'$ and $p_2 = p_2'$.

If $p_1 = c_1$, then, by (14) $x_a \neq a$, and so x_a-a-x-c_1-x_a is a hole of length four, a contradiction. So, from the symmetry, both p_1 and p_2 belong to P^*. By (2), the hole x-a-Q-x_a-p_1-x_b-Q-b-y-x is dominating, and since no vertex of P_2^* has a neighbour in it, it follows that P_2^* is empty, and therefore p_2 is adjacent to c_2. Similarly, p_1 is adjacent to c_1, and therefore P has length three.

Since c_1 is anticomplete to $N(x) \setminus N(y)$ and a'-p_1-c_1-x-a' is not a hole of length four for any $a' \in N(x) \setminus N(y)$, it follows that p_1 is anticomplete to $N(x) \setminus N(y)$, and, from the symmetry, $\{p_1, p_2\}$ is anticomplete to $N(K) \setminus (C \cup K)$. Consequently $x_a \neq a$ and $x_b \neq b$. So x-a-x_a-p_1-x_b-b-y-x is a hole of length seven, and therefore $m \leq 7$.

If $m = 7$, then the holes x-a-x_a-p_1-x_b-b-y-x and x-a-x_a-p_2-x_b-b-y-x show that p_1 and p_2 belong to A_3. Each of c_1, c_2 is complete to $\{x, y\}$, and has a neighbour in A_3, so by (10) both c_1 and c_2 are in C_3.

If $m = 5$, then by (10) $c_1, c_2 \in B_{x,y} \cup C_1 \cup C_3$, and since by (14) $\{c_1, c_2\}$ is anticomplete to $N(K) \setminus (C \cup K)$, it follows that $c_1, c_2 \in B_{x,y}$. Since $\{p_1, p_2\}$ is anticomplete to $N(K) \setminus (C \cup K)$, it follows that $p_1, p_2 \notin A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_3 \cup B_{1,2} \cup B_{2,3}$. But now, since

$$V(G) = W \cup \bigcup_{i=0}^{m-2} (B_i \cup C_i \cup B_{i,i+1}) \cup B_{x,y}.$$

and p_1, p_2 are anticomplete to $\{x, y\}$, we deduce that p_1, p_2 belong to B_2. This proves (15).

(16) Let $1 \leq j < i \leq m-2$ and let $p \in C_j$ be adjacent to $b \in B_i \cup B_{i-1,i}$. Then i is even, $b \in B_{i-1,i}$ and either

- *$i > j+1$, p is complete to A_j, and b is complete to $A_{i-1} \cup A_i$, or*
- *$i = j+1$ and $N(b) \cap A_{i-1} = N(p) \cap A_{i-1}$.*

Let a_i be a neighbour of b in A_i and let Q be a y-path from a_i. Then a_i-Q-y-p-b-a_i is a hole, and therefore Q has even length and so i is even.

By (10), j is odd. Let A'_j be the set of neighbours of p in A_j, and A'_{i-1} the set of neighbours of b in A_{i-1}. By (11) a_i is complete to A'_{i-1}. Let P be a path from A_{i-1} to A_j such that $|V(P) \cap A_k| = 1$ for $j \leq k \leq i-1$. Let $a_j \in A_j$ and $a_{i-1} \in A_{i-1}$ be the ends of P. We claim that $a_j \in A'_j$ if and only if $a_{i-1} \in A'_{i-1}$. Suppose first that $a_j \in A'_j$ and $a_{i-1} \in A_{i-1} \setminus A'_{i-1}$, and let $a \in A_i$ be adjacent to a_{i-1}. By (11), b is adjacent to a. But now p-a_j-P-a_{i-1}-a-b-p is an even hole, a contradiction. Next suppose that $a_j \in A_j \setminus A'_j$ and $a_{i-1} \in A'_{i-1}$, and let S be an x-path for a_j. Then p-x-S-a_j-P-a_{i-1}-b-p is an even hole, again a contradiction. This proves the claim. The claim implies that $A'_{i-1} \neq \emptyset$, and in particular, $b \in B_{i-1,i}$. If $i = j+1$, the claim implies that $N(b) \cap A_{i-1} = N(p) \cap A_{i-1}$, and (16) follows. So we may assume that $i > j+1$.

Assume that $A'_{i-1} \neq A_{i-1}$. Let $a'_{i-1} \in A'_{i-1}$ and $a_{i-1} \in A_{i-1} \setminus A'_{i-1}$. Let R' and R be x-paths for a'_{i-1} and a_{i-1}, respectively, and for $1 \leq k \leq i-2$ let a_k and a'_k be the vertices of R and R' in A_k, respectively. By the claim, $a'_j \in A'_j$ and $a_j \in A_j \setminus A'_j$; and by (5) and the claim, $V(R) \setminus (\bigcup_{k=0}^{j-1} A_k)$ is disjoint from $V(R') \setminus (\bigcup_{k=0}^{j-1} A_k)$, and for $j+1 \leq k \leq i-1$, a'_k is non-adjacent to a_{k-1} and a_k to a'_{k-1}. Consequently, since both R and R' can be completed to holes via y-paths for a_{i-1} and a'_{i-1}, respectively, (2) implies that for $j < k < i-1$, a_k is adjacent to a'_k. We recall that Q is a y-path for a_i. Since $i > j+1$, x-R-a_j-a_{j+1}-a'_{j+1}-R'-a'_{i-1}-a_i-Q-y-x is a hole of length $m+1$, and therefore even, a contradiction. This proves that $A'_{i-1} = A_{i-1}$. Since the claim implies that for every vertex $a_j \in A_j \setminus A'_j$, if T is a y-path for a_j then $V(T) \cap A_{i-1} \subseteq A_{i-1} \setminus A'_{i-1}$, it follows that $A_j = A'_j$.

Finally suppose that there exists a vertex $a \in A_i$ non-adjacent to b. Let S be an x-path for a and T a y-path, let a_{i-1} be the vertex of S in A_{i-1} and let a_j be the vertex of S in A_j. Then the paths p-b-a_{i-1}, p-a_j-S-a_{i-1} and p-y-T-a-a_{i-1} form a theta, contrary to 2.1. This proves that b is complete to A_i and completes the proof of (16).

(17) A_1 is complete to C_1.

Let $p \in C_1$. Since K is non-dominating in $G \setminus \{p\}$, it follows from (8) that p has a neighbour $b \in V(G) \setminus N(K)$. From the definition of C_1, we deduce that either $b \in B_1$, or $b \in B_i \cup B_{i-1,i}$ for some $2 \leq i \leq m-2$.

Assume first that $b \in B_1$. Then for all $a_1 \in A_1$, b-a_1-x-p-b is not a hole of length four, and therefore p is complete to A_1, as required. So we may assume that $b \in B_i \cup B_{i-1,i}$ for some $2 \leq i \leq m-2$, and by (16), p_1 is complete to A_1. This proves (17).

(18) Let $i \in \{1, \ldots, m-2\}$. If A_i contains an x-pair u, v, then A_i is complete to $A_{i-1} \cap N(u)$, $B_{i,i+1}$ is anticomplete to $\{u, v\}$, B_i is empty and $A_{i-1} \cap N(u)$ is complete to $B_{i-1,i}$.

The first assertion of (18) follows from (7). Since u-w-v-b-u is not a hole of length four, where $w \in N(u) \cap A_{i-1}$ and $b \in B_{i-1,i} \cup B_i$, it follows that $B_i = \emptyset$, and no vertex in $B_{i,i+1}$ is adjacent to both u and v.

Next suppose there exists $b \in B_{i,i+1}$ with a neighbour in $\{u, v\}$. From the symmetry we may assume that b is adjacent to u. Let a be a neighbour of u in A_{i+1}, and a' a neighbour of v in A_{i+1}. Since u, v is an x-pair, (6) implies that a is non-adjacent to v and a' is non-adjacent to u. By (11), b is non-adjacent to a', and therefore, again by (11), b is adjacent to v. But then b is adjacent to both u and v, a contradiction.

Finally, suppose that there exist $a \in A_{i-1} \cap N(u)$ and $b \in B_{i-1,i}$ non-adjacent. By (11), b is adjacent to both u and v. But now u-b-v-a-u is a hole of length four, a contradiction. This proves (18).

(19) The sets $B_{x,y}, B_{0,1}, B_{1,2}, \ldots, B_{m-2,m-1}$ are pairwise anticomplete; the sets $B_0, \ldots, B_{m-1}$ are pairwise anticomplete; and for all $i \in \{0, \ldots, m-2\}$ and $j \in \{0, \ldots m-1\}$ with $j \neq i, i+1$, $B_{i,i+1}$ is anticomplete to B_j.

Let $0 \leq i < j \leq m-2$. It is enough to prove that $B_i \cup B_{i,i+1}$ is anticomplete to $B_{j+1} \cup B_{j,j+1}$, B_i is anticomplete to B_{i+1} and $B_{x,y}$ is anticomplete to $B_{i,i+1}$.

Assume for a contradiction that there exist adjacent $u \in B_i \cup B_{i,i+1}$ and $v \in B_{j+1} \cup B_{j,j+1}$. Let a_i be a neighbour of $u \in A_i$ and a_{j+1} a neighbour of v in A_{j+1}, and let P be an x-path for a_i and Q a y-path from a_{j+1}. Then x-P-a_i-u-v-a_{j+1}-Q-y-x is a hole, say H. Since $V(H) \not\subseteq W$, x, y are vertices of H and H is odd, it follows that H has length at least $m+2$, a contradiction.

If $b_i \in B_i$ is adjacent to $b_{i+1} \in B_{i+1}$, then, by (11), b_i-b_{i+1}-a_{i+1}-a_i-b_i is a hole of length four for every adjacent $a_i \in A_i$ and $a_{i+1} \in A_{i+1}$, a contradiction.

Finally, assume that $b \in B_{x,y}$ has a neighbour $b' \in B_{i,i+1}$. Let a_i and a_{i+1} be neighbours of b' in A_i and A_{i+1}, respectively. Let P be an x-path for a_i and Q a y-path for a_{i+1}. Then $G|(V(P) \cup V(Q) \cup \{b, b'\})$ is a prism or an even wheel, contrary to 2.1. This proves (19).

(20) A_i is a clique for every odd integer i with $3 \leq i \leq m-4$.

Suppose there exists an odd integer $i \in \{3, \ldots, m-4\}$ such that A_i is not a clique. Then $m \geq 7$. From the symmetry and by (7) we may assume that every pair of non-adjacent vertices in A_i is an x-pair.

Let $a_1 \in A_1$ be a vertex complete to C_1 (such a vertex exists by (17)), and let u, u' be an x-pair in A_i. By (9) there exists a path P from u to a_1 such that $|V(P) \cap A_j| = 1$ for all $1 \leq j \leq i$. Let a_{i-1} be the vertex of P in A_{i-1}. By (18) a_{i-1} is complete to A_i. Let

$$L = V(P) \setminus \{u\}, T = \{x\} \cup C_1, S = A_i,$$

and

$$R = \bigcup_{j=i+1}^{m-2} (A_j \cup B_j \cup C_j \cup B_{j,j+1}) \cup B_{i,i+1} \cup B_{x,y} \cup \{y\}.$$

Then L is connected, anticomplete to R, the vertex $a_{i-1} \in L$ is complete to S and $L \setminus \{a_{i-1}\}$ is anticomplete to S, T is a clique by (12), and a_1 is complete to T. Let G' be the graph obtained from $G|(R \cup S \cup T)$ by adding all edges between S and T. Then, since i is odd, 2.3 implies that G' is even-hole-free. Since $i < m-2$, K is non-dominating in G', and therefore the minimality of $|V(G)|$ implies that there exists a vertex $v \in V(G') \setminus N_{G'}(K)$ that is bisimplicial in G'.

Next we show that v belongs to $V(G) \setminus N(K)$ and is bisimplicial in G, thus obtaining a contradiction. Let $R' = V(G') \setminus N_{G'}(K)$. Then

$$R' = \bigcup_{j=i+1}^{m-3} A_j \cup \bigcup_{j=i}^{m-3} B_j \cup B_{j,j+1} \cup B_{m-2},$$

and in particular $R' \subseteq V(G) \setminus N(K)$. By (18) B_i is empty, and consequently, by (19), R' is anticomplete to $V(G) \setminus V(G')$, and hence $N_G(v) = N_{G'}(v)$. Since, by (18), no vertex of $B_{i,i+1}$ is complete to A_i, (16) implies that C_1 is anticomplete to $B_{i,i+1}$, and therefore no vertex of R' has both a neighbour in S and a neighbour in T. It follows that G and G' induce the same graph on $N_{G'}(v)$, and therefore v is bisimplicial in G, a contradiction. This proves (20).

(21) Let $i \in \{4, \ldots, m-3\}$ be an even integer. If A_i does not contain a y-pair, then some vertex of A_{i-1} is complete to it. If A_{i-2} does not contain an x-pair then some vertex of A_{i-1} is complete to A_{i-2}. If A_i does not contain a y-pair and A_{i-2} does not contain an x-pair then some vertex of A_{i-1} is complete to $A_{i-2} \cup A_i$.

Assume that A_i does not contain a y-pair. We claim that some vertex of A_{i-1} is complete to A_i. Suppose not and let $a_{i-1} \in A_{i-1}$ be a vertex with a maximal set of neighbours in A_i. Let $N = A_i \cap N(a_{i-1})$. Then there exists $a'_i \in A_i \setminus N$. Let $a'_{i-1} \in A_{i-1}$ be a neighbour of a'_i. Now it follows from the choice of a_{i-1}, that there exists $a_i \in N \setminus N(a'_{i-1})$. But by (18) A_i is a clique, and by (20) A_{i-1} is a clique, and therefore a_i is adjacent to a'_i and a_{i-1} to a'_{i-1}. Consequently a_{i-1}-a_i-a'_i-a'_{i-1}-a_{i-1} is a hole of length four, a contradiction. This proves that some vertex $a \in A_{i-1}$ is complete to A_i. From the symmetry, if A_{i-2} contains no x-pair, then some vertex $a' \in A_{i-1}$ is complete to A_{i-2}. This proves the first two statements of (21).

Now assume that A_i does not contain a y-pair and A_{i-2} does not contain an x-pair, and let a, a' be as above. We claim that either a is complete to A_{i-2} or a' is complete to A_i. Suppose not, and let $a_i \in A_i \setminus N(a')$ and $a_{i-2} \in A_{i-2} \setminus N(a)$. Let P be an x-path for a_{i-2} and let Q be a y-path for a_i. By (20), a is adjacent to a'. By (5) x-P-a_{i-2}-a'-a-a_i-Q-y-x is a hole. But this hole has length $m+1$, and therefore it is even, a contradiction. This proves (21).

Let $\mathcal{P} = \bigcup_{j=1}^{m-2} C_j$. By (4), $C = \mathcal{P} \cup B_{x,y}$. For $1 \leq i \leq m-2$ let A'_i be the set of vertices of A_i with a neighbour in $\mathcal{P}$, and let $B'_i, B'_{i,i+1}$ be defined similarly. For $a \in A_{i-1}$ let $M(a) = N(a) \cap A_i$, $M'(a) = N(a) \cap A'_i$, $Q(a) = N(a) \cap B_{i-1,i}$ and $Q'(a) = N(a) \cap B'_{i-1,i}$; and and for $b \in B_{i-1,i}$ let $M(b) = N(b) \cap A_i$ and $M'(b) = N(b) \cap A'_i$.

(22) Let i be an odd integer such that $3 \leq i \leq m-4$. Assume that A_{i-1} contains no y-pair. Choose $w \in A_{i-1}$ with $M'(w)$ maximal, and subject to that with $Q'(w)$ maximal, and suppose that $B'_{i-1,i} \neq Q'(w)$. Then there exist $w' \in A_{i-1}$, $b, b' \in B'_{i-1,i}$ and $p \in C_i$ such that

1. *b-w-w'-b' is a path*

2. $M(w) = M(w') = M(b) = M(b') = A_i$

3. either $N(b) \cap \mathcal{P} \subseteq N(b') \cap \mathcal{P}$ *or* $N(b') \cap \mathcal{P} \subseteq N(b) \cap \mathcal{P}$*,*

4. p *is adjacent to both* b *and* b' *and complete to* A_i

Let b' be a non-neighbour of w in $B'_{i-1,i}$ and let w' be a neighbour of b' in A_{i-1}. (11) implies that b' is complete to $M(w)$, and so again by (11) w' is complete to $M(w)$. Now it follows from the choice of w that $Q'(w)$ is not a proper subset of $Q'(w')$, and therefore there exists $b \in Q'(w) \setminus Q'(w')$. Since A_{i-1} does not contain a y-pair, and since $M(w) \subseteq M(w')$, it follows that w is adjacent to w'. Since w-w'-b'-b-w is not a hole of length four, b is non-adjacent to b'. Thus b-w-w'-b' is a path and (22.1) holds.

By (11) and since b is non-adjacent to w', it follows that $M(w') \subseteq M(b)$ and similarly $M(w) \subseteq M(b')$. Again by (11), $M(b) \subseteq M(w)$ and $M(b') \subseteq M(w')$, and therefore all the inclusions hold with equality, that is $M(w) = M(w') = M(b) = M(b')$.

Next we claim that either $N(b) \cap \mathcal{P} \subseteq N(b') \cap \mathcal{P}$ or $N(b') \cap \mathcal{P} \subseteq N(b) \cap \mathcal{P}$. Suppose not, and let $p \in (N(b) \setminus N(b')) \cap \mathcal{P}$ and $p' \in (N(b') \setminus N(b)) \cap \mathcal{P}$. If p is non-adjacent to p', then, since i is odd and therefore A_{i-1} is anticomplete to $\mathcal{P}$ by (11), it follows that p-b-w-w'-b'-p' is a path with interior in $V(G) \setminus N(\{x, y\})$, contrary to (15), so p is adjacent to p'. But then p-b-w-w'-b'-p'-p is a hole of length six, a contradiction. This proves that (22.3) holds, and in particular, because $b, b' \in B'_{i-1,i}$, and therefore $N(b) \cap \mathcal{P} \neq \emptyset$ and $N(b') \cap \mathcal{P} \neq \emptyset$, there exists $p \in \mathcal{P}$ adjacent to both b and b'.

Since $\{b, b'\}$ is complete to $M(w)$, and b-a-b'-p-b is not a hole of length four for any $a \in M(w)$, it follows that p is complete to $M(w)$, and $p \in C_i$.

We claim that $M(b) = A_i$. For suppose not, and let $a \in A_i$ be non-adjacent to b, and therefore non-adjacent to b'. Let v be a neighbour of a in A_{i-1}. By (11) both b and b' are adjacent to v, and therefore b-v-b'-p-b is a hole of length four, a contradiction. This proves that $M(b) = A_i$, and hence $M(b) = M(b') = M(w) = M(w') = A_i$. So (22.2) and (22.4) follow. This proves (22).

(23) Let i *be an odd integer such that* $3 \leq i \leq m - 4$*, and* A_{i-1} *does not contain a* y*-pair, and* B_i *is empty. Then there is a vertex in* A_{i-1} *complete to* $A'_i \cup B'_{i-1,i}$*.*

First we claim that no vertex of C_i is complete to A_i. For assume for a contradiction that $p \in C_i$ is complete to A_i. By (15), since $m \geq 7$ and p is either complete or anticomplete to A_3, it follows that p is complete to $C \setminus \{p\}$. Let

$$S = \{y\} \cup A_i \cup C$$

$$T = \{x\} \cup \bigcup_{j=1}^{i-2} (A_j \cup B_j) \cup \bigcup_{j=0}^{i-1} B_{j,j+1}$$

and

$$U = \bigcup_{j=i+1}^{m-2} (A_j \cup B_j) \cup \bigcup_{j=i}^{m-2} B_{j,j+1}.$$

Then p is complete to $S \setminus \{p\}$. Since B_i is empty, $V(G) = S \cup T \cup U$, and by (19) T is anticomplete to U. But now S is a star cutset with centre p, and $x \notin S$, contrary to 4.1. This proves that no vertex of C_i is complete to A_i.

Choose $w \in A_{i-1}$ with $M'(w)$ maximal, and subject to that with $Q'(w)$ maximal. Suppose first that $A'_i \not\subseteq M'(w)$, and let $a \in A'_i$ be a non-neighbour of w in A'_i. Since by (20) A_i is a clique, it follows that a is complete to $M(w)$. Let p be a neighbour of a in C_i. Then p is not complete to A_i; let $a' \in A_i$ be a non-neighbour of p. Let w' be a neighbour of a' in A_{i-1}, choosing a' and w' so that $w' = w$ if possible, and let R' be an x-path for w'. Since x-R'-w'-a'-a-p-x is not an even hole, it follows that w' is adjacent to a, and therefore $w' \neq w$, and so p is complete to $M(w)$. But now, since x-R'-w'-a'-m-p-x is not an even hole for any $m \in M(w)$, it follows that w' is complete to $M(w) \cup \{a\}$, contrary to the choice of w. This proves that w is complete to A'_i. Finally, suppose that w is not complete to $B'_{i-1,i}$. Let w', b, b' and p be as in (22). But then p is complete to A_i, a contradiction. This proves (23).

(24) Let i be an odd integer such that $3 \leq i \leq m-4$ and A_{i-1} contains no y-pair. Then some vertex of A_{i-1} is complete to $B'_{i-1,i}$.

Suppose no such vertex exists. By (23) B_i is non-empty. By (22) there exist $w_1, w_2 \in A_{i-1}$, $b_1, b_2 \in B'_{i-1,i}$ and $p \in C_i$ such that

1. b_1-w_1-w_2-b_2 is a path
2. $M(w_1) = M(w_2) = M(b_1) = M(b_2) = A_i$
3. $N(b_2) \cap \mathcal{P} \subseteq N(b_1) \cap \mathcal{P}$
4. p is adjacent to both b_1 and b_2 and complete to A_i.

Since A_{i-1} contains no y-pair, and so, we deduce from (18) that $\{w_1, w_2\}$ is complete to $A_{i-1} \setminus \{w_1, w_2\}$. Let R_1 and R_2 be x-paths for w_1 and w_2, respectively.

We claim that there exist s, t with $\{s, t\} = \{1, 2\}$ and a path Q from b_s to a vertex of $B_{i,i+1}$ such that $V(Q)$ is anticomplete to $\{w_t, b_t\}$ and $Q^* \subseteq B_{i-1,i} \cup B_i$.

Let

$$U = \{x, y\} \cup C \cup \bigcup_{j=i+1}^{m-2} (A_j \cup B_j) \cup \bigcup_{j=i}^{m-2} B_{j,j+1}$$

$$S_1 = N(\{b_1, w_1\})$$

and

$$S_2 = N(\{b_2, w_2\}).$$

By 4.3 S_2 is not a double star cutset in G, and therefore there exists a path Q_2 from b_1 to a vertex $u \in U$ with $V(Q_2) \cap S_2 = \emptyset$, and such that $(V(Q_2) \setminus \{u\}) \cap U = \emptyset$. Since by (19) $B_{i-1,i} \cup B_i$ is anticomplete to $V(G) \setminus (U \cup S_2)$, it follows that $V(Q_2) \setminus \{u\} \subseteq B_{i-1,i} \cup B_i$, and therefore, again by (19), $u \in B_{i,i+1} \cup C$.

We may assume that $u \in C$, for otherwise the claim holds with $s = 1$ and $Q = Q_2$. Let Q' be a subpath of Q_2 with ends u, q' such that q' is adjacent to w_1 and no other vertex of Q' is. Let H be the hole w_1-q'-Q'-u-x-R_1-w_1. Then H is not dominating in G because $b_2 \in V(G) \setminus N(H)$ (since

$V(Q_2) \cap S_2 = \emptyset$). Let F be the component of $V(G) \setminus N(H)$ containing b_2. By 3.1 there is a vertex v in F that is bisimplicial in $G|(F \cup N(H))$, and therefore in G. Since there is no bisimplicial vertex of G in $V(G) \setminus N(K)$, we deduce that v is adjacent to y. Let T be a path from b_2 to v with $V(T) \subseteq F$. Since $A_{i-1} \cup A_i \cup B_{i-1} \subseteq N(w_1) \subseteq N(H)$, (19) implies that T contains a vertex of U. Let Q_1 be a minimal subpath of T containing b_2 and a vertex u' of U. Since $V(Q_1) \cap N(w_1) = \emptyset$, it follows that $u' \in B_{i,i+1} \cup C$, and since $V(Q_1) \subseteq F$, and in particular $\{x\}$ is anticomplete to $V(Q_1)$, we deduce that $u' \notin C$. But now the claim holds with $s = 2$ and $Q = Q_1$. This proves the claim.

Let Q be a path from b_s to a vertex u of $B_{i,i+1}$ with $V(Q) \setminus \{u\} \subseteq (B_{i-1,i} \cup B_i) \setminus S_t$ as in the claim. Let a be a neighbour of u in A_{i+1} and let T be a y-path for a. Let q' be the neighbour of w_s in Q such that the subpath Q' of Q between q' and u contains no other neighbour of w_s. Then x-R_s-w_s-q'-Q'-u-a-T-y-x is a hole and by (19) and the choice of Q', b_t has no neighbour in it, contrary to (2). This proves (24).

(25) $m < 9$.

Suppose $m \geq 9$ and let i be an even integer in $\{4, \ldots, m-5\}$. From the symmetry we may assume that A_i contains no y-pair. By (21) there exists a vertex $a \in A_{i-1}$ complete to A_i. Let P be an x-path for a.

Let

$$S = A_i, \;\; T = \{y\} \cup C, \;\; L = V(P)$$

and

$$R = (\bigcup_{j=i}^{m-2} A_j \cup B_j \cup B_{j,j+1}) \setminus A_i.$$

Then L is connected, anticomplete to R, by (15) T is a clique, T is complete to $\{x\}$, S is complete to a and anticomplete to $L \setminus \{a\}$. Let G' be the graph with $V(G') = S \cup T \cup R$, in which $u, v \in V(G')$ are adjacent if and only if there is an odd path between them with interior in L. By 2.3 G' is even-hole-free.

If $B_{i+1} = \emptyset$, let t be a vertex in A_i complete to $A'_{i+1} \cup B'_{i,i+1}$, and if $B_{i+1} \neq \emptyset$, let t be a vertex in A_i complete to $B'_{i,i+1}$ (the existence of such a vertex t follows from (23) and (24)). Then, since i is even, y is complete in G' to A_i, and in particular, y is adjacent in G' to t. Let $K' = \{y, t\}$. Then K' is anticomplete to A_{m-3}, and therefore K' is a non-dominating clique in G'. By the minimality of $|V(G)|$, there exists a vertex $v \in V(G') \setminus N_{G'}(K')$ that is bisimplicial in G'. Since y is complete to $A_i \cup A_{m-2} \cup B_{m-2,m-1} \cup C$ in G' and t is complete to B_i in G', it follows that

$$v \in \bigcup_{j=i+1}^{m-3} A_j \cup \bigcup_{j=i+1}^{m-2} B_j \cup \bigcup_{j=i}^{m-3} B_{j,j+1},$$

and hence by (5) and (19) $N_G(v) = N_{G'}(v)$. Let $N = N_G(v)$.

Since $N_G(x) \cap V(G') \subseteq C \cup \{y\}$ and $V(G') \setminus N_{G'}(y) \subseteq V(G) \setminus N_G(y)$, it follows that $v \in V(G) \setminus N_G(K)$, and therefore v is not bisimplicial in G. Consequently, $G|N \neq G'|N$, and so, from the construction of G' and since $y \notin N$, we deduce that $N \cap A_i \neq \emptyset$ and $N \cap C \neq \emptyset$. By (5) and (19), and since v is anticomplete to K' in G', this means that $v \in A_{i+1} \cup B_{i,i+1}$.

We claim that $N \cap C \subseteq \mathcal{P}$. Let $p \in N \cap C$. If $v \in A_{i+1}$, then $p \in C_{i+1}$ by the definition of C_{i+1}, and if $v \in B_{i,i+1}$, then $p \notin B_{x,y}$ by (19), and therefore $p \in \mathcal{P}$. This proves the claim.

Since v is non-adjacent to t and has a neighbour in $\mathcal{P}$, it follows from the choice of t that $v \in A_{i+1}$ and $B_{i+1} \neq \emptyset$. But now, let $a_i \in A_i$ and $a_{i+2} \in A_{i+2}$ be neighbours of v. Choose $b \in B_{i+1}$. Then a_i, a_{i+2} and b all belong to $N_{G'}(v)$, and they are pairwise non-adjacent, contrary to the fact that v is bisimplicial in G'. This proves (25).

(26) If $m = 7$ then C is a clique.

Suppose not and let $c_1, c_2 \in C$ be non-adjacent. By (1) there exists a path P between c_1 and c_2 with interior in $V(G) \setminus N(K)$, and by (15) $V(P) \cup K$ is dominating, P has length three, and $P^* \subseteq A_3$ and $c_1, c_2 \in C_3$. Since $V(P) \cup K$ is dominating, (16) implies $B_1 \cup B_{1,2} \cup B_2 \cup B_4 \cup B_{4,5} \cup B_5 = \emptyset$. By (1) and (19) $B_{0,1} = B_{5,6} = \emptyset$. Suppose $C_1 \neq \emptyset$, and let $p \in C_1$. By (1) and (16), p has a neighbour $b \in B_{3,4}$, and b is complete to A_3. By (15), c_1 has a neighbour $a \in A_3$, and c_1 is not complete to A_3. Therefore, by (16), it follows that b is non-adjacent to c_1. By (15), p is adjacent to c_1. But now, c_1-a-b-p-c_1 is a hole of length four, a contradiction. This proves that $C_1 = \emptyset$, and, from the symmetry, $C_5 = \emptyset$. Next suppose that there exists $p \in B_{x,y}$. By (1), p has a neighbour $d \in V(G) \setminus N(K)$, and so $d \in B_{2,3} \cup B_3 \cup B_{3,4}$, and from the symmetry we may assume that $d \in B_{2,3} \cup B_3$. Let a_3 be a neighbour of d in A_3, and let Q be a y-path for a_3. But now y-p-d-a_3-Q-y is a hole of length six, a contradiction. So $B_{x,y} = \emptyset$, and $N(K) = K \cup A_1 \cup A_5 \cup C_3$.

For $i = 1, 2$ let $D_i = N(c_i) \cap A_3$. Since c_1-x-c_2-a-c_1 is not a hole of length four for any $a \in A_3$, it follows that $D_1 \cap D_2 = \emptyset$. If there exist non-adjacent $d_1 \in D_1$ and $d_2 \in D_2$, then by (6) d_1 and d_2 have a common neighbour $a \in A_2 \cup A_4$, and x-c_1-d_1-a-d_2-c_2-x is a a hole of length six, a contradiction. So D_1 is complete to D_2.

We claim that $(A_2 \cup A_3 \cup A_4) \setminus (D_1 \cup D_2)$ is complete to $D_1 \cup D_2$. For let $a \in (A_2 \cup A_3 \cup A_4) \setminus (D_1 \cup D_2)$ have a non-neighbour $d \in D_1 \cup D_2$. From the symmetry we may assume that $d \in D_2$. By (20) A_3 is a clique, and hence $a \in A_2 \cup A_4$, and from the symmetry we may assume that $a \in A_4$. Choose $d_1 \in D_1$. Then c_1-d_1-d-c_2 is a path, and so, by (15, the set $\{x, y, c_1, c_2, d_1, d\}$ is dominating, and therefore a is adjacent to d_1. But now let a_5 be a neighbour of a in A_5. Then by (5) d-c_2-y-a_5-a-d_1-d is a hole of length six, a contradiction. This proves that $(A_2 \cup A_3 \cup A_4) \setminus (D_1 \cup D_2)$ is complete to $D_1 \cup D_2$.

Choose $d_1 \in D_1$. We claim that d_1 is complete to $B_{2,3} \cup B_{3,4}$. Suppose not; from the symmetry we may assume that $B' = B_{2,3} \setminus N(d_1)$ is non-empty. Since by (19) $N(B_{2,3}) \subseteq B_{2,3} \cup A_2 \cup A_3 \cup B_3 \cup C_3$, d_1 is complete to $A_2 \cup A_3 \cup B_3$ and $N(d_1)$ is not a full star-cutset in G by 4.2, it follows that some $b \in B'$ has a neighbour $c \in C_3 \setminus N(d_1)$. Since b-c_1-d_1-a_2-b is not a hole of length four for any $a_2 \in A_2$ adjacent to b, it follows that b is non-adjacent to c_1 . If c is non-adjacent to c_1, then b-c-x-c_1-d_1-a_2-b is a hole of length six, for any $a_2 \in A_2$ adjacent to b; and hence c is adjacent to c_1. Since c_1-c-d_2-d_1-c_1 is not a hole of length four for any $d_2 \in D_2$, c is anticomplete to D_2; and so by (16), b is anticomplete to D_2. Now, by the three previous sentences with the roles of c_1 and c_2 reversed, c is adjacent to c_2 and anticomplete to D_1. Since $c \in C_3$, there exists $d \in A_3$ adjacent to c, and $d \notin D_1$. But now, since A_3 is a clique, c_1-c-d-d_1-c_1 is a hole of length four, a contradiction. This proves that D_1 is complete to $B_{2,3} \cup B_{3,4}$.

Since K is a non-dominating clique in $G' = G \setminus \{d_1\}$, it follows from the minimality of $|V(G)|$ that some vertex $v \in V(G') \setminus N_{G'}(K)$ is bisimplicial in G'. Since G' is an induced subgraph of G

and no vertex of $V(G) \setminus N(K)$ is bisimplicial in G, it follows that $v \in V(G) \setminus N(K)$, v is adjacent to d_1 and d_1 has a non-neighbour n that is adjacent to v. Since $v \in N(d_1) \setminus N(K)$, it follows that $v \in A_2 \cup A_3 \cup A_4 \cup B_{2,3} \cup B_3 \cup B_{3,4}$. Let Q be a y-path for d_1. If $v \in B_{2,3} \cup B_3$, then $n \in C_3$, and d_1-Q-y-n-v-d_1 is a hole of length six, a contradiction. Similarly, $v \notin B_{3,4}$, and therefore $v \in A_2 \cup A_3 \cup A_4$.

Next assume that $v \in A_2$. Then $N(v) \subseteq A_1 \cup A_2 \cup A_3 \cup B_{2,3}$. Since v has a neighbour in A_1, and v is bisimplicial in G', it follows that $N(v) \cap A_2 = N_1 \cup N_2$ where $N_2 \cup (N_{G'}(v) \cap (A_3 \cup B_{2,3}))$ is a clique and $N_1 \cup (N_{G'}(v) \cap A_1)$ is a clique. But since d_1 is complete to $A_2 \cup A_3 \cup B_{2,3}$, it follows that $N_2 \cup (N_G(v) \cap (A_3 \cup B_{2,3})) = \{d_1\} \cup N_2 \cup (N_{G'}(v) \cap (A_3 \cup B_{2,3}))$ is a clique, and so v is bisimplicial in G, a contradiction. This proves that $v \notin A_2$, and from the symmetry $v \notin A_4$, and therefore $v \in A_3$ and $n \in C_3$. But now, choosing a_2 and a_4 to be neighbours of v in A_2 and A_4, respectively, we observe that a_2, a_4 and n are three pairwise non-adjacent neighbours of v in G', contrary to the fact that v is bisimplicial in G'. This proves (26).

Let S be a hole in G. We say that $v \in V(G) \setminus V(S)$ is a *centre* for S if v is complete to $V(S)$.

(27) Let S be a hole of length five with centre v. Let A be a connected subgraph of $V(G) \setminus (V(S) \cup \{v\})$, such that three consecutive vertices of S have neighbours in $V(A)$. Then v has a neighbour in $V(A)$.

Suppose not. We may assume that A is a minimal connected subgraph of $V(G) \setminus (V(S) \cup \{v\})$, such that three consecutive vertices of S have neighbours in $V(A)$. Let the vertices of S be $c_1, \ldots, c_5$ in order. Let P be a path between two non-consecutive vertices of S with $P^* \subseteq V(A)$ and with $|V(P)|$ minimum. Without loss of generality, we may assume that the ends of P are c_1 and c_3. Since v has no neighbour in P^* and c_1-P-c_3-v-c_1 is not an even hole, it follows that P is odd. Since c_1-P-c_3-c_4-c_5-c_1 is not an even hole, it follows that one of c_4, c_5 has a neighbour in P^*, and from the minimality of $|V(P)|$ and by the symmetry, we may assume that c_1 and c_5 have a common neighbour p in P^*, and p is the unique neighbour of c_5 in P^*. Suppose c_2 has a neighbour $p' \in P^*$. From the minimality of $|V(P)|$, p' is adjacent to c_3 and c_2 has no neighbour in $P^* \setminus \{p'\}$. But now c_1-P-p'-c_2-c_1 is an even hole, a contradiction. So c_2 has no neighbour in P^*. From the symmetry we deduce that c_4 has no neighbour in P^*. Let D be a minimal connected subgraph of A, such that $P^* \subseteq V(D)$, and at least one of c_2, c_4 has a neighbour in $V(D)$. If both c_2, c_4 have neighbours in D, then, since $\{c_2, c_4\}$ is anticomplete to P^*, the minimality of D implies that some $d \in V(D)$ is adjacent to both c_2 and c_4, and c_2-d-c_4-v-c_2 is a hole of length four, a contradiction. So we may assume that c_4 has a neighbour in $V(D)$, and c_2 does not. Let Q be a path in D from c_4 to a vertex $q \in V(D)$, such that q has a neighbour in P^*, and no other vertex of Q does. By the minimality of A, q has a unique neighbour p' in P^*, and p' is adjacent to c_3. If c_5 has a neighbour in $V(Q) \setminus \{c_4\}$, then $A \setminus \{p\}$ is a connected subgraph of $V(G) \setminus (V(S) \cup \{v\})$, and c_3, c_4, c_5 all have neighbours in $V(A \setminus \{p\})$, contrary to the minimality of A. So c_5 has no neighbour in $V(Q) \setminus \{c_4\}$. If c_1 has no neighbour in $V(Q)$, then c_4-Q-p'-P-p-c_5-c_4 and c_4-Q-p'-P-p-c_1-v-c_4 are two holes of different parity, and therefore one of them is even, a contradiction. So c_1 has a neighbour in $V(Q)$. Let q' be the neighbour of c_1 in $V(Q)$, such that the subpath of Q between q and q' contains no other neighbour of c_1. Since c_1-q'-c_4-v-c_1 is not a hole of length four, c_4 is non-adjacent to q'. But now, there exists a path T between c_1 and c_3 with interior in $V(q\text{-}Q\text{-}q') \cup \{p'\}$, and neither of c_4, c_5 has a neighbour in T^*. But now one of c_1-T-c_3-v-c_1 and c_1-T-c_3-c_4-c_5-c_1 is an even hole, a contradiction . This proves (27).

(28) Let $u \in V(G) \setminus N(K)$ and let $v \in V(G) \setminus (N(K) \cup \{u\})$ be a bisimplicial vertex of $G \setminus \{u\}$. Then $N_G(v) \setminus N_G(u)$ is not a clique.

Let $G' = G \setminus \{u\}$. Suppose $N_G(v) \setminus N_G(u)$ is a clique. Then, since v is a bisimplicial vertex of G', there is no stable set of size three in $N_G(v)$. Since v is not bisimplicial in G, it follows that u is adjacent to v and $G|(N_G(v))$ contains an odd antihole. Since every odd antihole of length at least seven contains a hole of length four, it follows that $G|(N_G(v))$ contains an antihole of length five, and therefore a hole of length five. Let S be such a hole. Since v is bisimplicial in G'. it follows that $u \in V(S)$. Let the vertices of S be a-u-b-b'-a'-a. Let F be the component of $V(G) \setminus N(v)$ containing $\{x, y\}$. By 4.2, $F = V(G) \setminus N(v)$. We claim that u has a neighbour in F, for otherwise $N_G(u) \setminus \{u\} \subseteq N_{G'}(v)$, and therefore u is bisimplicial in G and non-adjacent to both x and y, a contradiction. Let $A = N_G(v) \setminus (N_G(b') \cup \{u\})$, $B = N_G(v) \setminus (N_G(a') \cup \{u\})$ and $D = N_G(v) \cap N_G(a') \cap N_G(b')$. Then $a \in A$ and $b \in B$. Since v is bisimplicial in G', both A and B are cliques; and since $N_G(v) \setminus N_G(u)$ is a clique, it follows that $N_G(v) \setminus N_G(u) \subseteq D$, and so u is complete to $A \cup B$.

By (27), not both a and b have neighbours in F, and from the symmetry we may assume that a does not. If a has a neighbour $z \in B \setminus A$, then a-a'-b'-z-a is a hole of length four, and therefore a is anticomplete to $B \setminus A$. Thus $N(a) \subseteq A \cup D \cup \{a', u, v\}$. Since a-d-b'-d'-a is not a hole of length four, where $d, d' \in D \cap N(a)$, it follows that $D \cap N(a)$ is a clique. But now, $N_G(a)$ is the union of two cliques, namely $A \cup \{u, v\}$, and $N(a) \cap (D \cup \{a'\})$. Since a is anticomplete to F, it follows that a is a bisimplicial vertex of G in $V(G) \setminus N(K)$, a contradiction. This proves (28).

(29) $m = 5$.

Suppose $m > 5$. By (25), $m = 7$. By (26) C is a clique. Assume first that A_2 does not contain a y-pair. Let $R = \bigcup_{i=3}^{5} A_i \cup B_i \cup B_{i,i+1}$, let $C' = (C \cup \{y\}) \setminus (C_1 \cup \{x\})$, and let G' be the graph obtained from $G|(A_2 \cup C' \cup C_1 \cup R)$ by adding all the edges between A_2 and C'. By (17), C_1 is complete to A_1. We claim that G' is even-hole-free.

Assume first that A_2 contains an x-pair. By (18), there exists a vertex $a_1 \in A_1$ complete to A_2. Let $S = A_2$, $T = C' \cup C_1$ and $L = \{a_1, x\}$. Then L is connected, anticomplete to R, T is complete to x and S is complete to a_1 and anticomplete to x. Since no vertex of $T \setminus C_1$ is adjacent to a_1, and since C_1 is complete to A_1, 2.3 implies that G' is even-hole-free.

Next assume that A_2 is a clique. Suppose there is an even hole H is G'. Since G is even-hole-free, it follows that $V(H) \cap A_2 \neq \emptyset$, and $V(H) \cap C' \neq \emptyset$. Since $A_2 \cup C'$ is a clique in G', it follows that $|V(H) \cap A_2| = |V(H) \cap C'| = 1$. Let $V(H) \cap A_2 = \{h_1\}$ and $V(H) \cap C' = \{h_2\}$. Then h_1 is adjacent to h_2 in G', and $G|V(H)$ is a path from h_1 to h_2, say P, with interior in $R \cup C_1$. Since H is an even hole in G', P is odd. Let a_1 be the neighbour of h_1 in A_1. Since h_1-P-h_2-x-a_1-h_1 is not an even hole, it follows that one of a_1, x has a neighbour in P^*, and therefore some vertex of P^* is in C_1. Since C is a clique, it follows that the neighbour h_3 of h_2 in P belongs to C_1, and no other vertex of P does. But now h_1-P-h_3-a_1-h_1 is an even hole, a contradiction. This proves that if A_2 is a clique, then G' is even-hole-free, and completes the proof of the claim.

If $B_3 = \emptyset$, let $a_2 \in A_2$ be as in (23), and if $B_3 \neq \emptyset$, let $a_2 \in A_2$ be as in (24). Let $U = \{y, a_2\}$. Since U is anticomplete to A_4, U is a non-dominating clique in G', and therefore, by the minimality of $|V(G)|$, there exists a vertex $v \in V(G') \setminus N_{G'}(U)$ that is bisimplicial in G'. Since y is complete to $A_2 \cup C_1$, and a_2 is complete to $B_2 \cup C'$, it follows from (5) and (19) that $v \in R$, and therefore $N_G(v) = N_{G'}(v)$. Since v is not bisimplicial in G, it follows that $G|(N_G(v)) \neq G'|(N_{G'}(v))$, and therefore in G, v has both a neighbour in A_2 and a neighbour in C'. This, together with (19), implies that $v \in A'_3 \cup B'_{2,3}$. From the choice of a_2 and the fact that v is non-adjacent to a_2, we deduce that $v \in A'_3$ and $B_3 \neq \emptyset$. But then v has a neighbours in A_2, A_4, B_3, and these are three pairwise non-adjacent neighbours of v in G', contrary to the fact that v is bisimplicial in G'.

This proves that A_2 contains a y-pair, and, therefore, no x-pair. From the symmetry, it follows that A_4 contains an x-pair and no y-pair. By (18), this implies that $B_2 = B_4 = \emptyset$.

Let a_3 in A_3 be complete to $A_2 \cup A_4$ (such a vertex a_3 exists by applying (21) with $i = 4$). By (20) A_3 is a clique, by (18) a_3 is complete to $B_{2,3} \cup B_{3,4}$, and since $B_2 \cup B_4 = \emptyset$, we deduce that a_3 is complete to

$$A_2 \cup A_3 \cup A_4 \cup B_2 \cup B_3 \cup B_4 \cup B_{2,3} \cup B_{3,4}.$$

Let $G' = G \setminus \{a_3\}$. Since K is a non-dominating clique in G', (8) implies that there exists $v \in V(G') \setminus N_{G'}(K)$, adjacent to a_3 and bisimplicial in G'. Since v is adjacent to a_3, we deduce that $v \in A_2 \cup A_3 \cup A_4 \cup B_3 \cup B_{2,3} \cup B_{3,4}$. From the symmetry we may assume that $v \in A_2 \cup A_3 \cup B_3 \cup B_{2,3}$.

Assume first that $v \in A_3 \cup B_3 \cup B_{2,3}$. By (5), (18) and (19), it follows that $N(v) \setminus N(a_3)$ is a subset of C, and therefore $N(v) \setminus N(a_3)$ is a clique, contrary to (28). This proves that $v \notin A_3 \cup B_3 \cup B_{2,3}$, and therefore $v \in A_2$.

By (11), $N_{G'}(v) \subseteq A_1 \cup A_2 \cup A_3 \cup B_2 \cup B_{1,2} \cup B_{2,3}$. Suppose v has a neighbour b in $B_{1,2}$. Since every non-adjacent pair in A_2 is a y-pair, (18) implies that v is complete to $A_2 \setminus \{v\}$. Let a_2, a'_2 be a y-pair in A_2. Then, by (18), a_2, a'_2, b is a stable set of size three in $N_{G'}(v)$, contrary to the fact that v is bisimplicial in G'. This proves that v is anticomplete to $B_{1,2}$, and $N(v) \setminus N(a_3) \subseteq A_1$. But now, since v-n-x-n'-v is not a hole of length four for any $n, n' \in N(v) \cap A_1$, it follows that $N(v) \cap A_1$ is a clique, and we get a contradiction to (28). This proves (29).

In view of (29), from now on we assume that $m = 5$.

(30) Every vertex in $B_{x,y}$ is anticomplete to $B_1 \cup B_3 \cup B_{1,2} \cup B_{2,3}$, and every vertex of $B_{x,y}$ has a neighbour in B_2.

Let $b \in B_{x,y}$, and suppose that b has a neighbour $b' \in B_{1,2} \cup B_{2,3} \cup B_1 \cup B_3$. From the symmetry we may assume that $b' \in B_3 \cup B_{2,3}$. Choose $a \in A_3$ adjacent to b'. Now y-b-b'-a-y is a hole of length four, a contradiction. This proves that $B_{x,y}$ is anticomplete to $B_1 \cup B_3 \cup B_{1,2} \cup B_{2,3}$. By (8), every vertex of $B_{x,y}$ has a neighbour in $V(G) \setminus N(K)$, and therefore every vertex of $B_{x,y}$ has a neighbour in B_2. This proves (30).

(31) A_2 is a clique.

Suppose not. Then by (18) $B_2 = \emptyset$ and we may assume that A_2 contains a y-pair, and therefore by (6) no x-pair. Let a_2, a'_2 be two non-adjacent vertices in A_2 and let $a_1 \in A_1$ be adjacent to a_2 and not a'_2, and $a'_1 \in A_1$ adjacent to a'_2 and not a_2.

Let us say that a'_2 *dominates* a_2 if there exists a path P in G, from a'_2 to a vertex p, adjacent to one of x, y, and such that $\{a_1, a_2\}$ is anticomplete to $V(P)$, and $V(P) \cap C_3 = \emptyset$. We claim that either a_2 dominates a'_2, or a'_2 dominates a_2.

By 4.3, the edge $a'_1a'_2$ is not the centre of a double star cutset D such that K and a_2 are in different component of $V(G) \setminus D$, and therefore there exists a path P in G, from a_2 to a vertex p, adjacent to one of x, y, and such that $\{a'_1, a'_2\}$ is anticomplete to $V(P)$. We may assume that a_2, a'_2 are chosen such that $|V(P)|$ is minimum. If $V(P) \cap C_3 = \emptyset$, then a_2 dominates a'_2, and the claim holds, so we may assume that $p \in V(P) \cap C_3$ and $V(P) \setminus \{p\}$ is anticomplete to $\{x, y\}$. We claim that a_1 is anticomplete to $V(P)$. Since $V(P) \setminus \{p\}$ is anticomplete to $\{x, y\}$, it follows that $V(P) \cap (A_1 \cup B_{0,1} \cup C_1) = \emptyset$. By (11) and (18), $B_1 \cup B_{1,2}$ is complete to $\{a_1, a'_1\}$ and so, since a'_1 is anticomplete to $V(P)$, it follows that $V(P) \cap (B_1 \cup B_{1,2}) = \emptyset$. By the minimality of $|V(P)|$, it follows that $V(P) \cap A_2 = \{a_2\}$. By (30) and since $B_2 = \emptyset$, it follows that $B_{x,y} = \emptyset$. But now, since

$$N(a_1) \subseteq A_1 \cup C_1 \cup B_1 \cup B_{0,1} \cup B_{1,2} \cup A_2 \cup B_{x,y} \cup \{x\},$$

we deduce that a_1 is anticomplete to $V(P)$. Let H_0 be the hole x-a_1-a_2-P-p-x. By 4.4, $G \setminus (N(V(H_0) \setminus \{y\})$ is connected. Therefore, there is a path P' in G, from a'_2 to a vertex p', adjacent to y, and such that $V(H_0)$ is anticomplete to $V(P')$. In particular, since by (15) C_3 is a clique, it follows that $V(P') \cap C_3 = \emptyset$, and $\{a_1, a_2\}$ is anticomplete to $V(P')$. This proves the claim.

By the claim and from the symmetry, we may that a'_2 dominates a_2. Choose such P as in the claim, of minimum length, over all choices of the vertex $a'_2 \in A_2$, anticomplete to $\{a_1, a_2\}$. Then $V(P) \setminus \{p\}$ is anticomplete to $\{x, y\}$. Since by (11) every vertex of $B_1 \cup B_{1,2}$ is adjacent to one of a_1, a_2, it follows that $P \cap (B_1 \cup B_{1,2}) = \emptyset$. From the minimality of P, and since $\{a_1, a_2\}$ is anticomplete to $V(P)$, it follows that $V(P) \cap A_2 = \{a'_2\}$.

By (30) $B_{x,y} = \emptyset$, since $B_2 = \emptyset$. By (17), C_1 is complete to a_1, and therefore $V(P) \cap (B_{x,y} \cup C_1) = \emptyset$. Suppose $V(P) \cap (A_1 \cup B_{0,1}) \neq \emptyset$. It follows from the minimality of P, that the only vertex of P in $A_1 \cup B_{0,1}$ is p. Let p' be the neighbour of p in P. If $p \in A_1$ then $N(p) \setminus N(K) \subseteq A_2 \cup B_1 \cup B_{1,2}$; and if $p \in B_{0,1}$ then, by (19), $N(p) \setminus N(K) \subseteq B_1$. Since $p' \notin N(K) \cup B_1 \cup B_{1,2}$, it follows that $p' = a'_2$, $p \in A_1$, and p is adjacent to a'_2, and not to a_1 or a_2, contrary to (6). This proves that $V(P) \cap (A_1 \cup B_{0,1}) = \emptyset$, and consequently

$$V(P) \subseteq \{a'_2\} \cup A_3 \cup B_3 \cup B_{2,3} \cup B_{3,4}.$$

Thus $p \in A_3 \cup B_{3,4}$, and $V(P) \setminus \{p\} \subseteq \{a'_2\} \cup B_3 \cup B_{2,3}$. But now let H_1 be the hole a'_2-P-p-y-x-a'_1-a'_2. Then $x, y \in V(H_1)$ and $a_2 \notin N(H_1)$, contrary to (2). This proves (31).

(32) C is a clique.

Suppose not, and let $c_1, c_2 \in C$ be non-adjacent. By (15), $c_1, c_2 \in B_{x,y}$, and every path P from c_1 to c_2 with interior in $V(G) \setminus N(K)$ satisfies $P^* \subseteq B_2$. For $i = 1, 2$, let $N_i = B_2 \cap N(c_i)$. By (30), both N_1 and N_2 are non-empty. (15) implies that N_1 and N_2 are disjoint. If some $n_1 \in N_1$ and $n_2 \in N_2$ are non-adjacent, then for every $a_2 \in A_2$ the path c_1-n_1-a_2-n_2-c_2 contradicts (15), so N_1 is complete to N_2. Since c_1-n_1-n_2-n'_1-c_1 is not a hole of length four for $n_1, n'_1 \in N_1$ and $n_2 \in N_2$, it follows that N_1, and similarly N_2, is a clique. Also by (15), for every $n_1 \in N_1$ and $n_2 \in N_2$, the set $\{c_1, c_2, n_1, n_2, x, y\}$ is dominating. By (19) and (30), this implies that $B_1 = B_3 = \emptyset$. Since by (8) and (19) every vertex in $B_{0,1}$ has a neighbour in B_1, $B_{0,1} = \emptyset$, and similarly $B_{3,4} = \emptyset$. Since $\{c_1, c_2, n_1, n_2, x, y\}$ is dominating, (30) implies that every vertex $b \in B_{1,2} \cup B_{2,3}$ is adjacent to at least one of n_1, n_2. If $b \in B_{1,2}$ is adjacent to n_1 and not n_2, then x-a_1-b-n_1-n_2-c_2-x is a hole of length six, where a_1 is a neighbour of b in A_1, a contradiction. So $B_{1,2}$ is complete to $\{n_1, n_2\}$, and since n_1, n_2 were chosen arbitrarily, it follows that $B_{1,2}$ is complete to $N_1 \cup N_2$. From the symmetry, $B_{2,3}$ is also complete to $N_1 \cup N_2$.

Fix $n_1 \in N_1$. Suppose some vertex $b \in B_2$ is non-adjacent to n_1. Then $b \notin N_1$, and so b is non-adjacent to c_1. We claim that every neighbour of b in $B_{x,y}$ is adjacent to c_1. For suppose not, and let c be a neighbour of b in $B_{x,y}$ non-adjacent to c_1. Then the path c_1-n_1-a_2-b-c contradicts (15), where $a_2 \in A_2$. This proves the claim. Let M be a component of $B_2 \setminus N(n_1)$ containing b. By the previous argument applied to any vertex of M instead of b, we deduce that $N(M) \cap B_{x,y} \subseteq N(c_1)$. Let

$$X = A_2 \cup B_{1,2} \cup B_{2,3} \cup C_1 \cup C_3 \cup N_1 \cup N_2 \cup (N(c_1) \cap B_{x,y}).$$

By (19), $N(M) \subseteq X$. But now, since $A_2 \cup B_{1,2} \cup B_{2,3} \cup N_1 \cup N_2 \subseteq N(n_1)$, and by (15), $C_1 \cup C_3 \subseteq N(c_1)$, X is a double star cutset that contradicts 4.3. This proves that $B_2 \setminus \{n_1\}$ is complete to n_1.

Let $G' = G \setminus \{n_1\}$. Since K is a non-dominating clique in G, by (8) some vertex $v \in N(n_1) \setminus N(K)$ is bisimplicial in G'. Consequently, $v \in B_{1,2} \cup B_{2,3} \cup A_2 \cup B_2$. Since every vertex in A_2 has three pairwise non-adjacent neighbours in G', namely n_2, some $a_1 \in A_1$ and some $a_3 \in A_3$, we deduce that $v \notin A_2$. From the symmetry we may assume that $v \in B_2 \cup B_{1,2}$. But now, by (19), $N(v) \setminus N(n_1) \subseteq C_1 \cup C_3 \cup B_{x,y} \cup A_1$, and in particular $N(v) \setminus N(n_1)$ is complete to x. Since x-u-v-u'-x is not a hole of length four for $u, u' \in N(v) \setminus N(n_1)$, we deduce that $N(v) \setminus N(n_1)$ is a clique, contrary to (28). This proves (32).

(33) Let $a_2, a_2' \in A_2$ and $b \in B_{1,2}$ such that b-a_2-a_2' is a path and let P be a path from a neighbour of a_2' in $B_{1,2}$ to a vertex with a neighbour in K, such that $\{a_2, b\}$ is anticomplete to $V(P)$, and only one vertex of P has a neighbour in K. Then $V(P) \subseteq A_1 \cup B_1 \cup B_{1,2} \cup B_{0,1} \cup C$ and $V(P) \cap C \neq \emptyset$.

Let p be the unique vertex of P with a neighbour in K. Then p is an end of P. Since by (31) A_2 is a clique, and A_2 is complete to B_2, it follows that $V(P) \cap (A_2 \cup B_2) = \emptyset$. Since $A_1 \cup B_1 \cup B_{1,2} \cup B_{0,1}$ is anticomplete to $A_3 \cup B_3 \cup B_{2,3} \cup B_{3,4}$, $(V(P) \setminus \{p\}) \cap (C \cup A_2 \cup B_2 \cup K) = \emptyset$, and $V(P) \cap B_{1,2} \neq \emptyset$, it follows that $V(P) \subseteq A_1 \cup B_1 \cup B_{1,2} \cup B_{0,1} \cup C$.

It remains to prove that $p \in C$. Suppose not, and choose P of minimum length, violating (33). By the minimality, the first vertex, b', of P is in $B_{1,2} \cap N(a_2')$ and no other vertex of $V(P) \cap B_{1,2}$ is adjacent to a_2'. Since p has a neighbour in $\{x, y\}$, we deduce that $p \in A_1 \cup B_{0,1}$. Let Q be a y-path for a_2'. Then x is anticomplete to $V(Q) \setminus \{y\}$, and y is anticomplete to $V(P)$, and therefore there exists a hole H such that $x, y \in V(H)$ and $V(H) \subseteq V(P) \cup V(Q) \cup \{x\}$. By the choice of P, b is anticomplete to $V(P)$, and since $b \in B_{1,2}$, b is anticomplete to $V(Q) \cup \{x\}$. But this means that b is anticomplete to $V(H)$, contrary to (2). This proves (33).

(34) Some vertex of A_2 is complete to $B_{1,2}$.

Suppose not. For a vertex $a \in A_2$ let $M(a) = N(a) \cap B_{1,2}$. Let $a_2 \in A_2$ be a vertex with $M(a_2)$ maximal. Then there exists $b' \in B_{1,2} \setminus M(a_2)$. Let a_2' be a neighbour of b' in A_2. By the choice of a_2, there exists a vertex $b \in B_{1,2}$ adjacent to a_2 and not to a_2'. By (31), a_2 is adjacent to a_2', and since a_2-b-b'-a_2'-a_2 is not a hole of length four, b is non-adjacent to b'.

Since by 4.3, the edge $a_2'b'$ is not the centre of a double star-cutset D such that b and K are in different components of $V(G) \setminus D$, it follows that there exists a path S from b to a vertex with a neighbour in K such that $\{a_2', b'\}$ is anticomplete to $V(S)$. By choosing b appropriately, we may assume a_2 is anticomplete to $(V(S) \setminus \{b\}) \cap B_{1,2}$. We may also assume that only the last vertex, c, of S has a neighbour in K. By (33), $V(S) \subseteq A_1 \cup B_1 \cup B_{1,2} \cup B_{0,1} \cup C$ and $c \in C$. Let T be a y-path for a_2. Then y is anticomplete to $V(S) \setminus c$. If c has a neighbour in $V(T) \setminus \{y\}$, then $c \in C_3$, the neighbour s of c in S is in $B_1 \cup B_{1,2}$, and c-x-a_1-s-c is a hole of length four, where $a_1 \in A_1$ is a neighbour of s, a contradiction. This proves that c is anticomplete to $V(T) \setminus \{y\}$. Therefore there exists a hole H, with $c, y \in V(H)$ and $V(H) \subseteq V(S) \cup V(T)$; and b' has no neighbour in $V(H)$. Since A_3 is anticomplete to $V(S) \setminus \{c\}$, it follows that $a_2 \in V(H)$. If a_2 has a neighbour in $n \in V(S) \setminus B_{1,2}$ then $n \in A_1$, and b' is anticomplete to $\{a_2, n\}$, contrary to (11). This proves that a_2 is anticomplete to $V(S) \setminus B_{1,2}$, and, consequently, $b \in V(H)$. Now, by 4.4, $V(G) \setminus N(V(H) \setminus \{y\})$ is connected, and we deduce that there exists a path P from b' to a vertex p with a neighbour in $\{x, y\}$, such that $N(V(H) \setminus \{y\})$ is anticomplete to $V(P)$. Since $a_2, b \in V(H)$ and $c \in V(H) \cap C$, and, by (32), C is a clique, it follows that $V(P) \cap C = \emptyset$, contrary to (33). This proves (34).

(35) $B_1 \cup B_{0,1} = \emptyset$.

Suppose $B_1 \neq \emptyset$. Let $S = C_1 \cup B_{x,y} \cup \{x\}$, $T = A_2$, $R = A_1 \cup B_1 \cup B_{0,1} \cup B_{1,2}$ and $L = A_3 \cup \{y\}$. Then S is a clique by (32) and T is a clique by (31), L is connected, S is complete to y and anticomplete to $L \setminus \{y\}$, every vertex of T has a neighbour in L, and L is anticomplete to R. Let G' be the graph obtained from $G|(R \cup S \cup T)$ by adding all the edges st with $s \in S$ and $t \in T$. By 2.3 G' is even-hole-free. Let $a_2 \in A_2$ be a vertex complete to $B_{1,2}$ (such a vertex exists by (34)). Since $B_1 \neq \emptyset$, $\{x, a_2\}$ is a non-dominating clique in G', and the minimality of $|V(G)|$ implies that there exists $v \in V(G') \setminus N_{G'}(\{a_2, x\})$ that is bisimplicial in G'. Since S and T are both cliques, x is complete to $A_1 \cup B_{0,1}$ and a_2 is complete to $B_{1,2}$, it follows that $v \in B_1$. Since v is not a bisimplicial vertex of G, and v is anticomplete to T, it follows that v has a neighbour $u \in V(G) \setminus V(G')$. By (19) $u \in C$, and since $C = C_1 \cup C_3 \cup B_{x,y}$ and $u \notin S$, it follows that $u \in C_3$. But now v-a_1-x-u-v is a hole of length four, for every $a_1 \in A_1$, a contradiction. This proves that $B_1 = \emptyset$.

Next assume that $B_{0,1} \neq \emptyset$ and choose $b \in B_{0,1}$. By (8), b has a neighbour u in $V(G) \setminus N(\{x, y\})$. But (19) implies that $u \in B_1$, a contradiction. This proves (35).

From (35) and the symmetry we deduce that $B_3 \cup B_{3,4} = \emptyset$, and so

$$V(G) = A_1 \cup A_2 \cup A_3 \cup B_{1,2} \cup B_{2,3} \cup B_2 \cup C_1 \cup C_3 \cup B_{x,y} \cup \{x, y\}.$$

(36) No vertex in A_2 *is complete to* $B_{1,2} \cup B_{2,3}$.

Suppose such a vertex u exists, and choose u with a maximal set of neighbours in $A_1 \cup A_3$, over all vertices of A_2 complete to $B_{1,2} \cup B_{2,3}$.

We claim that K is a non-dominating clique in the graph $G \setminus \{u\}$. Suppose the claim is false. Then $B_2 = B_{1,2} = B_{2,3} = \emptyset$, $A_2 = \{u\}$, and u is complete to $A_1 \cup A_3$. By (7), this implies that A_1 and A_3 are both cliques, and therefore u is a bisimplicial vertex in G, a contradiction. This proves that K is a non-dominating clique in the graph $G \setminus \{u\}$.

Let N_1 and N_3 be the sets of neighbours of u in A_1 and A_3, respectively. By the minimality of $|V(G)|$ and since $\{x, y\}$ is non-dominating in the graph $G' = G \setminus \{u\}$, there exists a vertex $v \in V(G) \setminus (N(K) \cup \{u\})$ such that $v \in N_G(u)$ and v is a bisimplicial vertex of G'. Since $v \in V(G) \setminus (N(K) \cup \{u\})$, it follows that $v \in A_2 \cup B_2 \cup B_{1,2} \cup B_{2,3}$, and from the symmetry we may assume that $v \in A_2 \cup B_2 \cup B_{1,2}$ and therefore $N_G(v) \setminus N_G(u) \subseteq C \cup A_1 \cup A_3$. Since by (32) C is a clique, it follows that v has a neighbour in $A_1 \cup A_3$, and therefore $v \in A_2 \cup B_{1,2}$.

Assume first that $v \in B_{1,2}$. Then $N_G(v) \setminus N_G(u) \subseteq C \cup A_1$, and therefore $N_G(v) \setminus N_G(u)$ is complete to x. Let $n_1, n_2 \in N_G(v) \setminus N_G(u)$. Since x-n_1-v-n_2-x is not a hole of length four, it follows that n_1 is adjacent to n_2, and therefore $N_G(v) \setminus N_G(u)$ is a clique, contrary to (28). This proves that $v \in A_2$.

Since $v \in A_2$, it follows that v is anticomplete to C. Since A_1 contains an x-pair, and therefore, by (7), no y-pair, it follows that $N(v) \cap A_1$ is a clique, and similarly, $N(v) \cap A_3$ is a clique. Since by (28) $N_G(v) \setminus N_G(u) \subseteq A_1 \cup A_3$ is not a clique, we deduce that v has a neighbour $a_1 \in A_1 \setminus N(u)$ and a neighbour $a_3 \in A_3 \setminus N(u)$. If v has a non-neighbour in $b \in B_{1,2}$, then by (11), b is adjacent to a_1, and u-v-a_1-b-u is a hole of length four, a contradiction. So v is complete to $B_{1,2}$, and similarly, v is complete to $B_{2,3}$. From the choice of u it now follows that there exists a vertices $a'_1 \in A_1$, adjacent to u and not to v. But now x-a'_1-u-v-a_3-y-x is a hole of length six, a contradiction. This proves (36).

By (34), there exist vertices $v_1, v_3 \in A_2$ such that v_1 is complete to $B_{1,2}$ and v_3 to $B_{2,3}$; choose v_1 and v_3 with maximal sets of neighbours in A_1 and A_3, respectively. Let N_1 be the set of neighbours of v_1 in A_1, and N_3 the set of neighbours of v_3 in A_3. By (32) v_1 is adjacent to v_3. If there exist $n_1 \in N_1$ and $n_3 \in N_3$ such that n_1 is non-adjacent to v_3, and n_3 is non-adjacent to v_1, then x-n_1-v_1-v_3-n_3-y-x is a hole of length six, a contradiction. Consequently, from the symmetry we may assume that v_3 is complete to N_1. By (36), v_3 is not complete to $B_{1,2}$. Since by 4.2 v_3 is not the centre of a full star cutset in G, there is a path from a vertex of $B_{1,2} \setminus N(v_3)$ to one of x, y, containing no neighbour of v_3. Since by (11), $N(B_{1,2}) \cap A_1 \subseteq N_1$, and by (19) $B_{1,2}$ is anticomplete to $B_{2,3}$, and v_3 is complete to $B_2 \cup A_2 \setminus \{v_3\}$, it follows that there exist an edge bc with $b \in B_{1,2} \setminus N(v_3)$ and $c \in C \setminus N(v_3)$. Since b-a_1-x-c-b is not a hole of length four, where $a_1 \in A_1 \cap N(b)$, it follows that $c \in C_1 \setminus N(v_3)$. Since if some $n_3 \in N_3$ is non-adjacent to v_1, then y-c-b-v_1-v_3-n_3-y is a hole of length six, we deduce that v_1 is complete to N_3. But now, from the symmetry, there exists an edge $b'c'$ with $b' \in B_{2,3} \setminus N(v_1)$ and $c' \in C_3 \setminus N(v_1)$, and since by (32) c is adjacent to c', c-b-v_1-v_3-b'-c'-c is a hole of length six, a contradiction. This completes the proof of 5.1.

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