Vertex-Colouring Edge-Weightings

L. Addario-Berry[a], K. Dalal[a], C. McDiarmid[b], B. A. Reed[a] and A. Thomason[c]

[a]School of Computer Science, McGill University, University St.
Montreal, QC, H3A 2A7, Canada

[b]Department of Statistics, University of Oxford, 1 South Parks Road,
Oxford OX1 3TG, United Kingdom

[c]DPMMS, Centre for Mathematical Sciences, Wilberforce Road,
Cambridge CB3 0WB, United Kingdom

Abstract

A weighting w of the edges of a graph G induces a colouring of the vertices of G where the colour of vertex v, denoted c_v, is $\sum_{e \ni v} w(e)$. We show that the edges of every graph that does not contain a component isomorphic to K_2 can be weighted from the set $\{1, \ldots, 30\}$ such that in the resulting vertex-colouring of G, for every edge (u, v) of G, $c_u \neq c_v$.

1. Introduction

A *k-edge-weighting* of a graph G is an assignment of an integer weight, $w(e) \in \{1, \ldots, k\}$ to each edge e. The edge-weighting is *proper* if no two edges incident to the same vertex receive the same weight. It is *vertex-injective* if for every pair of distinct vertices, $\{u, v\}$, the colours c_u and c_v are distinct, where the colour of a vertex is defined as the sum of the weights on the edges incident to that vertex. It is *vertex-colouring* if for every edge (u, v), the colours c_u and c_v are distinct.

An edge-weighting is *adjacent vertex-distinguishing* if for every edge (u, v), the multiset of weights appearing on edges incident to u is distinct from the multiset of weights appearing on edges incident to v. It is *vertex-distinguishing* if the above condition holds for all pairs of distinct vertices u and v. Proper adjacent vertex-distinguishing edge-weightings and proper vertex-distinguishing edge-weightings have been studied by many researchers [2,3] and are reminiscent of harmonious colourings (see [5]). If a k-edge-weighting is vertex-colouring then it is adjacent vertex-distinguishing, though the converse may not hold. On the other hand, an adjacent vertex-distinguishing edge-weighting using real weights $\{w_1, \ldots, w_k\}$ which are linearly independent over $\mathcal{Q}$ is also vertex-colouring.

Clearly a graph cannot have a vertex-colouring or adjacent vertex-distinguishing edge-weighting if it has a component which is isomorphic to K_2 i.e., an *edge component*. In [7], Karoński, Łuczak and Thomason initiated the study of vertex-colouring and adjacent vertex-distinguishing edge-weightings i.e., where the edge-weighting is not necessarily proper. They conjectured that every graph without an edge component permits a vertex-colouring 3-edge-weighting. Among other results, they proved their conjecture for the case of 3-colourable graphs (see Theorem 1 of [7]). Additionally, they showed that every graph without an edge complonent permits an adjacent vertex-distinguishing 213-edge-weighting and that graphs with minimum degree at least 10^{99} permit an adjacent vertex-distinguishing 30-edge-weighting. Recently, [1] showed that every graph without an edge component permits an adjacent vertex-distinguishing 4-edge-weighting and that graphs of minimum degree at least 1000 permit an adjacent vertex-distinguishing 3-edge-weighting. We complement these results by proving the following theorem:

Theorem 1.1 Every graph without an edge component permits a vertex-colouring 30-edge-weighting.

We shall prove Theorem 1.1 in Section 3. In order to do so, some preliminary results will be useful.

2. Preliminary Results

We shall need the following two theorems for our main result.

Theorem 2.1 Given a connected, non-bipartite graph G, a set of target colours t_v for all $v \in V(G)$, and an integer k, where k is odd or $\sum_{v \in V} t_v$ is even, there exists a k-edge-weighting of G such that for all $v \in V(G)$, $c_v \equiv t_v \pmod{k}$.

Proof Idea The proof is an easy modification of the proof of Theorem 1 from [7]; here's a sketch. The conditions mean there exists a number m such that $2m \equiv \sum t_v \pmod{k}$. Put weight m on one edge and zero on the rest. Since the sum of the vertex weights is now correct, if one vertex has a wrong weight so does another. In this case choose an even length walk between these vertices and alternately add/subtract the same number from the edge weights along the walk; such a walk always exists in a connected non-bipartite graph. The sum of vertex weights is preserved, while the number of correct vertex weights increases.

If H is a spanning subgraph of G, or if W is a subset of $V = V(G)$, then $d_H(v)$ and $d_W(v)$ will have their natural meanings; that is, $d_H(v)$ denotes the number of neighbours of v in H and $d_W(v)$ denotes $|N(v) \cap W|$ where $N(v)$ is the set of neighbours of v in G.

Theorem 2.2 Suppose that for some graph F we have chosen, for every vertex v, two integers a_v^- and a_v^+ such that $\frac{d(v)}{3} - 1 \leq a_v^- \leq \frac{d(v)}{2}$ and $\frac{d(v)}{2} - 1 \leq a_v^+ \leq \frac{2d(v)}{3}$. Then there is a spanning subgraph H of F such that for every vertex v:

$d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$.

The proof for Theorem 2.2 relies on the following lemma, a modification of lemmas which appear in [8,6].

Lemma 2.3 Let G be a graph and suppose we have chosen, for each vertex v, non-negative integers a_v and b_v such that $a_v < b_v$. Then, precisely one of the following holds:

(i) G has a spanning subgraph H such that $\forall v$, $a_v \leq d_H(v) \leq b_v$, or

(ii) $\exists\ A, B \subset V(G)$, $A \cap B = \emptyset$ such that:

$$\sum_{v \in A} (a_v - d_{V-B}(v)) > \sum_{v \in B} b_v.$$

Furthermore, if (ii) holds, we can find A and B satisfying (ii) such that for all $w \in A$, we can choose a subgraph H of G such that for all v, $d_H(v) \leq b_v$, which minimizes

$$\sum_{v \in V} \max(0, a_v - d_H(v))$$

and satisfies:

(I) $d_H(v) \leq a_v \quad \forall\ v \in A.$

(II) $d_H(v) = b_v \quad \forall\ v \in B.$

(III) $d_H(w) < a_w.$

Proof. Clearly at most one of (i) and (ii) can hold, since otherwise the number of H edges from A to B would exceed the number from B to A.

Choose H such that $d_H(v) \leq b_v$ for all v and minimizing $\sum_{v \in V} \max(0, a_v - d_H(v))$. If this sum is zero, then (i) holds. Otherwise, the set $A' = \{v \mid d_H(v) < a_v\}$ is non-empty. By an H-*alternating path* we mean a path in G with one endpoint in A' whose first edge is not in H and whose edges alternate between being in H and not being in H.

We let A be those vertices which are the endpoints of some H-alternating path of even length (thus $A' \subseteq A$ since we permit paths of length 0). We let B be those vertices which are the endpoints of some H-alternating path of odd length.

For any H-alternating path P of length > 0, we let H_P be the graph with $E(H_P) = E(H - P) + E(P - H)$. Then, H_P contradicts our choice of H unless either P is odd and one endpoint v of P satisfies $d_H(v) = b_v$, or P is even and both endpoints v of P satisfy $d_H(v) \leq a_v$. This implies (I) and (II).

Thus, A and B are disjoint. Furthermore, by the definition of A and B, $\forall\ e$ with one endpoint in A and no endpoint in B, $e \in H$ and $\forall\ e$ with one endpoint in B and no endpoint in A, $e \notin H$. Thus,

$$\sum_{v \in A} d_H(v) = \sum_{v \in B} b_v + \sum_{v \in A} d_{V-B}(v),$$

which combined with the fact that $\sum_{v \in A} a_v > \sum_{v \in A} d_H(v)$ implies (ii).

To see that (III) also holds, if $w \in A - A'$, let P be an H-alternating path ending in w and replace H by H_P. This proves the lemma. ∎

Remark. See Lovasz and Plummer ([9], Chapter 10), or Schrijver ([10], Volume A, Section 35.1) for two proofs of a similar theorem which inspired our result.

Proof of Theorem 2.2 For each vertex v, choose either $a_v = a_v^-$, $b_v = a_v^- + 1$ or $a_v = a_v^+$, $b_v = a_v^+ + 1$ and a subgraph H with $d_H(v) \le b_v$ for all v minimizing $\sum_{v \in V} \max(0, a_v - d_H(v))$ over all such choices of a_v and H. If Condition (i) of Lemma 2.3 holds, then we are done. Otherwise (ii) holds, and we construct A and B as in the proof of that lemma. We will need the following claims:

Claim 2.4 For all $v \in A$, $a_v - d_{V-B}(v) \le \frac{1}{2} d_B(v)$.

Claim 2.5 For all $v \in B$, $b_v \ge \frac{1}{2} d_A(v)$.

Given that these claims hold, we have

$$\sum_{v \in A} (a_v - d_{V-B}(v)) \le \frac{1}{2} \sum_{v \in A} d_B(v) = \frac{1}{2} \sum_{v \in B} d_A(v) \le \sum_{v \in B} b_v,$$

a contradiction. So it remains to prove our claims.

Proof of Claim 2.4 Let $v \in A$. We apply (III) to ensure $d_H(v) < a_v$. Without loss of generality, $a_v^- \le a_v^+$. Further, we can assume $a_v = a_v^+$ as otherwise

$$a_v - d_{V-B}(v) \le \frac{1}{2} d(v) - d_{V-B}(v) \le \frac{1}{2} d_B(v),$$

as desired. Also, $d_H(v) > a_v^- + 1$ as otherwise setting $a_v = a_v^-$ contradicts the fact that our choices minimized $\sum_{u \in V} \max(0, a_u - d_H(u))$.

More strongly, $d_{H-B}(v) > a_v^- + 1$ as otherwise setting $a_v = a_v^-$ and deleting $d_H(v) - a_v^- - 1$ edges of H between v and B contradicts our choice of H. Thus,

$$d_{V-B}(v) \ge d_{H-B}(v) > a_v^- + 1 \ge \frac{1}{3} d(v),$$

which implies:

$$d_B(v) < \frac{2}{3} d(v)$$

and hence

$$d_{V-B}(v) > \frac{1}{2} d_B(v).$$

So,

$$\begin{aligned} a_v - d_{V-B}(v) &\le \frac{2}{3} d(v) - d_{V-B}(v) \\ &= \frac{2}{3} d_B(v) - \frac{1}{3} d_{V-B}(v) \\ &< \frac{2}{3} d_B(v) - \frac{1}{6} d_B(v) = \frac{1}{2} d_B(v). \quad \blacksquare \end{aligned}$$

Proof of Claim 2.5 Let $v \in B$. We can assume that $b_v = a_v^- + 1 < \frac{1}{2}d(v)$ as otherwise $b_v \geq \frac{1}{2}d(v)$ and the claim holds. Note now that $a_v^+ > a_v^-$. Furthermore, $d_A(v) > 2b_v = 2(a_v^- + 1)$ or the claim holds. Thus, there is a vertex u of A joined to v by an edge not in H. As in the proof of Lemma 2.3, we can augment along an H-alternating path ending in u so as to ensure that $d_H(u) < a_u$ without changing $d_H(v)$, A, B or the fact that $uv \notin E(H)$. Now, we re-set $a_v = a_v^+$ and $b_v = a_v^+ + 1$ and then, choose a set S of $a_v^+ - a_v^- \geq 1$ vertices of A joined to v by an edge of $G - H$ *including* u. This is possible because $d_A(v) > 2(a_v^- + 1) \geq a_v^+$.

Now set $H' = H + \{wv \mid w \in S\}$. Let us check that these choices contradict the minimality of our original choices of the a_x and H. For each $x \in S$ we have $d_{H'}(x) = d_H(x) + 1 \leq a_x + 1 = b_x$; and for the vertex v we have $d_{H'}(v) = d_H(v) + a_v^+ - a_v^- = a_v^+ + 1 = b_v$. Thus $d_{H'}(x) \leq b_x$ for each $x \in V$, as required. Further, for each $x \in V$ including $x = v$ we have $\max(0, a_x - d_{H'}(x)) \leq \max(0, a_x - d_H(x))$, and $\max(0, a_u - d_{H'}(u)) = \max(0, a_u - d_H(u)) - 1$. Thus we have the contradiction we sought. ∎

3. The Proof of Theorem 1.1

Recall that a maximum 2-cut of V is a partition of V into V_1, V_2 that maximizes the number of edges between V_1 and V_2.

Definition *A recursive k-cut is defined as follows. As a base case, a recursive 1-cut of $V(G)$ is simply $V_1 = V(G)$. For $k > 1$, start with a recursive $(k-1)$-cut, $V_1, \ldots, V_{k-1}$. Let V'_{k-1} and V'_k be a maximum 2-cut of V_{k-1}. The recursive k-cut is then $V_1, V_2, \ldots, V_{k-2}, V'_{k-1}, V'_k$.*

For a recursive k-cut, it is easy to see that for all $v \in V_i$ with $i < k$,

$$\left| N(v) \cap \bigcup_{j>i} V_j \right| \geq |N(v) \cap V_i|,$$

by the property of the maximum 2-cut. In addition, for all $v \in V_k$, it is easy to see that

$$|N(v) \cap V_1| \geq 2^{k-2} |N(v) \cap V_k|. \tag{1}$$

Without loss of generality, assume that G is connected, and non-bipartite. (If G is bipartite then by Theorem 1 of [7] there exists a vertex-colouring 3-edge-weighting.) Consider a recursive 8-cut of the vertices of G into sets $V_1, V_2, \ldots, V_8$. If there are no internal edges in V_8, then we use Theorem 2.1 to weight the edges such that vertices in V_i have colour i mod 9 using weights $\{1, \ldots, 9\}$. We next process the sets $V_1, \ldots, V_7$ in order. For each set, process the vertices of V_i in an arbitrary order. (Recall that c_v is the sum of weights on the edges incident to v.) For each $v \in V_i$ in turn, we add 9 to a subset of the edges from v to $\bigcup_{j>i} V_j$ so that c_v differs from c_u for all previously processed $u \in N(v) \cap V_i$. This is possible since there are at least as many edges to $\bigcup_{j>i} V_j$ as there are neighbours in V_i. As this step does not change the arity mod 9 of any vertex, vertices in different sets have distinct colours. Also, for sets $V_1, \ldots V_7$, two adjacent vertices in the same set have different colours and since V_8 has no internal edges, the resulting 18-edge-weighting is vertex-colouring.

Before we handle the case where V_8 contains an internal edge, we sketch the idea. We again start with the recursive 8-cut. The edges are initially weighted such that adjacent vertices in different sets have distinct colours mod 12 with all the vertices in a particular set having the same specially chosen colour mod 12. We then add 12 to the weights on some of the edges between V_i and $\bigcup_{j>i} V_j$ for i from 1 to 7 in turn so that adjacent vertices within each set have different colours. As this does not change the arity of any vertex colour mod 12, all adjacent vertices in the first seven sets will then have different colours. To complete the construction, we consider the set of edges between V_1 and V_8 and assign either -3 or $+3$ to each of these edges to distinguish the remaining vertices.

Assume that there is at least one internal edge in V_8. Then, using Theorem 2.1, we weight the edges so that vertices in V_i have colours $0, 1, 2, 4, 7, 8, 10, 11$ mod 12 for $i = 1, 2, 3, 4, 5, 6, 7, 8$, respectively, allowing at most one specified vertex in V_7 to be coloured either 10 or 11 mod 12. This flexibility ensures that no parity problems arise. (Since there are internal edges in V_8, V_7 is not empty.) If this vertex ends up coloured 11 mod 12, then we denote it v^*. If v^* exists, it is moved into V_8. If any edge-weight is less than 4, we add 12 to it, maintaining the arity of every vertex. (This positive bias prevents negative edge-weights later in the construction. Thus, the edge-weights can range from 4 to 15 at this point.)

We next process the sets $V_1, \ldots, V_7$ in order. For each set, process the vertices of V_i in an arbitrary order. For each $v \in V_i$ in turn, we add 12 to a subset of the edges from v to $\bigcup_{j>i} V_j$ so that c_v differs from c_u for all previously processed $u \in N(v) \cap V_i$. This is possible since that are at least as many edges to $\bigcup_{j>i} V_j$ as there are neighbours in V_i. (NB: This remains true even if v^* exists since it was moved to V_8. Also, the edge-weights can range from 4 to 27 at this point.)

Finally, we need to process vertices in V_8. Consider $G' \subset G$ with the vertex set $V_1 \cup \{v \in V_8 | v \text{ has a neighbor in } V_8\}$, and the edge set consisting of the edges in G between V_1 and V_8 except that for any vertex in $V_8 \cap G'$ whose degree (with respect to G') is odd, we discard an incident edge arbitrarily. We will add either $+3$ or -3 on each of the edges of G' in such a way that we distinguish adjacent vertices in V_8. (A vertex v in $V_8 \setminus G'$ has no neighbours in V_8 and thus need not be considered further.) Note that adding -3 and 3 to the edge-weights implies that the arity of every vertex $v \in V_8$ will belong to the set $\{5, 11\}$ since $d_G'(v)$ is even. Further, the arity of the vertices in V_1 will be in the set $\{0, 3, 6, 9\}$. Thus the arities of any two sets do not intersect and since we will only change weights for edges in G', we only need to consider vertices in V_1 and V_8 to complete the construction.

Consider the set $T = V_7 \cup V_8$. Every vertex v in T has at least 32 $d_T(v)$ edges to V_1 by (1). In particular, for vertices $v \in V_8 \cap G'$ (which may include v^*),

$$d_{G'}(v) \geq 32 d_T(v) \geq 32 d_{V_8}(v).$$

Define the intervals, L_v, U_v as follows:

$$L_v = \left[\frac{d_{G'}(v)}{3} - 1, \frac{d_{G'}(v)}{2}\right], \qquad U_v = \left[\frac{d_{G'}(v)}{2} - 1, \frac{2d_{G'}(v)}{3}\right].$$

Using $|L_v|$ and $|U_v|$ to denote the number of integers in the intervals L_v and U_v respectively, we see that:

$$\begin{aligned} |L_v|, |U_v| &\geq \left\lfloor \frac{d_{G'}(v)}{6} \right\rfloor + 1 \\ &\geq \left\lfloor \frac{32 d_{V_8}(v)}{6} \right\rfloor + 1 \\ &\geq \left\lfloor \frac{24 d_{V_8}(v) + 6}{6} \right\rfloor + 1 \\ &= 4 d_{V_8}(v) + 2 \end{aligned}$$

for any $v \in V_8 \cap G'$ using the fact $d_{V_8}(v) \geq 1$.

Given $a_v^- \in L_v$, $a_v^+ \in U_v$, let $S_v = \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$. For vertices $v \in V_1$, we choose $a_v^- = a_v^+ = \left\lfloor \frac{d_{G'}(v)}{2} \right\rfloor$. (Recall that c_v is the sum of the weights on edges incident to v, before starting to process V_8.) Suppose that for all $v \in V_8$ we can find $a_v^- \in L_v$ and $a_v^+ \in U_v$ so that for all $u \in N(v) \cap V_8$ and for all $a_u \in S_u$ and $a_v \in S_v$, $c_u + 3\Delta_u(a_u)$ differs from $c_v + 3\Delta_v(a_v)$ where $\Delta_v(a) = a - (d_{G'}(v) - a)$. By Theorem 2.2, there exists a subgraph $H \subseteq G'$ such that for all v, $d_H(v) \in S_v$. Take such a subgraph H and add $+3$ to the weight of every edge in H and -3 to the weight of each edge in $G' - H$. Before this step, the colours of all vertices in V_1 were 0 mod 12 and adjacent vertices had different colours, so the colours of adjacent vertices must have differed by at least 12. Thus, after this step, two adjacent vertices of V_1 still have different colours since this step will add at most 6 and subtract at most 3 from any single vertex in V_1. Adjacent vertices of V_8 now have different colours by our choice of the $\{a_v\}$. Thus, the resulting weights are between 1 and 30 and induce a vertex-colouring of G.

It remains to explain how we can find such a_v^- and a_v^+ for $v \in V_8$. Consider the vertices of V_8 in some arbitrary order. For each vertex $v \in V_8$, greedily choose integers $a_v^- \in L_v$, $a_v^+ \in U_v$ such that the following conditions hold:

a) $c_v + 3\Delta_v(a_v^-) \equiv c_v + 3\Delta_v(a_v^+) \equiv 11 \pmod{12}$.

b) $\{c_v + 3\Delta_v(a_v) | a_v \in S_v\} \cap \{c_u + 3\Delta_u(a_u) | a_u \in S_u\}$ is empty for any $u \in N(v) \cap V_8$ already considered,

where $\Delta_v(a) = a - (d_{G'}(v) - a)$. By our construction, c_v is 11 mod 12 and $d_{G'}(v)$ is even, thus Condition a) is equivalent to requiring that a_v^- and a_v^+ have the same parity as $\frac{d_{G'}(v)}{2}$. Note that we need only choose a_v^-, a_v^+ so that $c_v + 3\Delta_v(a_v^-)$ and $c_v + 3\Delta_v(a_v^+)$ are not equal to $c_u + 3\Delta_u(a_u^-)$ or $c_u + 3\Delta_u(a_u^+)$ for any $u \in N(v) \cap V_8$ already considered. Arity considerations then force the intersection in Condition b) to be empty.

As at most $d_{V_8}(v)$ neighbours of v have already been processed, there are at most $2d_{V_8}(v)$ integers which a_v^- (resp. a_v^+) must avoid. Since L_v (resp. U_v) contains at least $4d_{V_8}(v) + 2$ integers, there are at least $2d_{V_8}(v) + 1$ choices with the right parity, so a valid choice must exist. ∎

4. Conclusions and Open Problems

4.1. Strengthening Theorem 2.2

The proof of Theorem 1.1 relies heavily on our ability to find a *degree-constrained subgraph*: a spanning subgraph $H \subseteq G$ such that the degree of every vertex in H is in a specified set. This set is given by Theorem 2.2 and consists, for each vertex, of two pairs consecutive integers, each within a specific range. If Theorem 2.2 could be strengthened by increasing the size of these ranges, this would improve the upper-bound on the number of weights needed in a vertex-colouring edge-weighting.

The ranges in Theorem 2.2 are in essence $[x, 1/2]d(v)$ and $[1/2, (1-x)]d(v)$ for $x = 1/3$. The following construction shows that the theorem would fail to hold for $x < 1/6$. Let F be a K_{2n+1} and a $\overline{K_n}$ joined by all edges - thus F has $3n+1$ vertices. Choose $a_v^- = a_v^+ = \lfloor d(v)/2 \rfloor = n$ on the independent set. Choose $a_v^- = \lfloor xd(v) \rfloor = \lfloor 3nx \rfloor$, $a_v^+ = \lceil (1-x)d(v) \rceil - 1 = \lceil 3(1-x)n \rceil - 1$ on the clique. Suppose a spanning subgraph $H \subseteq F$ can be found as in the theorem - we will show that for $x < 1/6$ this implies that both conditions (i) and (ii) of Lemma 2.3 hold, which is impossible. Choose $a_v = d_H(v)$ if $d_H(v) \in \{a_v^-, a_v^+\}$ or $a_v = d_H(v) - 1$ otherwise, and $b_v = a_v + 1$. With these choices, condition (i) of the lemma holds. Let A be the set of vertices of K_{2n+1} for which $d_H(v) \in \{a_v^+, a_v^+ + 1\}$. Note that $F - H$ also satisfies the degree requirements theorem because for all v, the set $\{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$ is symmetric about $d(v)/2$ - thus we can assume $|A| \geq n+1$.

Set $r = |A|$ and $B = V - A$. Also, let

$$S_1 = \sum_{v \in A} a_v = \lceil 3n(1-x) - 1 \rceil \cdot r,$$

$$S_2 = \sum_{v \in B} b_v = \sum_{v \in \overline{K_n}} b_v + \sum_{v \in K_{2n+1} - A} b_v = n(n+1) + \lfloor 3nx + 1 \rfloor \cdot (2n+1-r), \text{ and}$$

$$S_3 = \sum_{v \in A} d_{V-B}(v) \leq (r-1)^2 \leq r^2 - 2n - 1.$$

This yields

$$S_2 + S_3 - S_1 < r^2 - 3nr + (6x+1)n^2 + 3n.$$

Letting $x = (1-\epsilon)/6$ for any small ϵ, we have

$$S_2 + S_3 - S_1 \leq (r-n)(r-2n) + 3n - \epsilon n^2 < 0$$

for $r \in [n+1, 2n]$ and n large enough, which implies condition (ii) also holds. The authors are as yet unsure what the correct value of x should be - it is possible that Theorem 2.2 is tight.

4.2. Louigi's Conjecture

Suppose we are given a graph G and, for each vertex v a list of acceptable degrees $D_v \subseteq \{0, 1, \ldots, d(v)\}$. How large do the lists D_v need be in order to guarantee that we can find a spanning subgraph H such that for every v, $d_H(v) \in D_v$? We now impose no constraints on the ranges of these degrees. It is thus plausible that we should need many more choices than in Theorem 2.2. The authors make the following conjecture:

Conjecture 4.1 *Given a graph $G = (V, E)$ and, $\forall v \in V$, a list $D_v \subseteq \{0, 1, \ldots, d(v)\}$ satisfying $|D_v| > \lceil d(v)/2 \rceil$, there is a spanning subgraph $H \subseteq G$ so that for all v, $d_H(v) \in D_v$.*

There are three different sorts of extreme example. Firstly, suppose that G is bipartite and all vertex degrees are even. Let $D_v = \{0, 1, \ldots, d(v)/2\}$ on one part of G and let $D_v = \{d(v)/2, \ldots, d(v)\}$ on the other part, except that for one vertex we delete the $d(v)/2$ from its set. Thus $|D_v| = d(v)/2 + 1$ for all but one vertex which does not have the $+1$, but counting edges shows that there can be no H as required. Secondly, let G be bipartite with all degrees odd. Let $D_v = \{0, 1, \ldots, \lfloor d(v)/2 \rfloor\}$ on one part of G and $D_v = \{\lceil d(v)/2 \rceil, \ldots, d(v)\}$ on the other part. Thus $|D_v| = \lceil d(v)/2 \rceil$ for all v but again, counting edges shows that there can be no H as required. A final example has all vertex degrees even, and $D_v = \{0, 2, \ldots, d(v)\}$, the $d(v)/2 + 1$ even numbers up to $d(v)$ for each vertex v, except for one vertex where we take the $d(v)/2$ such odd numbers. Again there can clearly be no H as required, since H would have exactly one vertex of odd degree.

It is interesting to note that Theorem 2.2 implies that the conjecture would hold if $|D_v| > \lceil d(v)/2 \rceil$ were replaced by $|D_v| > \lceil 11d(v)/12 \rceil$. Once the sets D_v are this large, we are guaranteed two consecutive choices in $[d(v)/3 - 1, d(v)/2]$ and in $[d(v)/2 - 1, 2d(v)/3]$.

4.3. Acknowledgements

The authors would like to thank the referee for a careful reading of the paper.

REFERENCES

1. L. Addario-Berry, R. E. L. Aldred, K. Dalal, B. A. Reed, Vertex Colouring Edge Partitions, *accepted to J. of Combinatorial Theory B*
2. M. Aigner, E. Triesch and Zs. Tuza, Irregular Assignments and Vertex-Distinguishing Edge-Colorings of Graphs, *Combinatorics 90*, (A. Barlotti et al, editors), Elsevier Science Pub., New York (1992) 1-9
3. A. C. Burris and R. H. Schelp, Vertex-Distinguishing Proper Edge Colourings, *Journal of Graph Theory 26 (2)* (1997) 73-82
4. P. N. Balister, O. M. Riordan, R. H. Schelp, Vertex-Distinguishing Edge Colorings of Graphs, *Journal of Graph Theory 42 (2)* (2003) 95-109
5. K. Edwards, The harmonious chromatic number of bounded degree graphs, *J. London Math. Soc. 2 55 (3)* (1997) 435-447
6. K. Heinrich, P. Hell, D. G. Kirkpatrick, G. Z. Liu, A Simple Existence Criterion for $(g < f)$-factors *Discrete Mathematics 85* (1990) 313-317
7. M. Karoński, T. Łuczak and A. Thomason, Edge weights and vertex colours, *J. Combinatorial Theory B 91* (2004) 151-157
8. L. Lovász, The factorization of graphs II, *Acta Math. Acad. Sci. Hungaricae 23* (1972) 223-246
9. L. Lovasz and M. D. Plummer, *Matching Theory*, Academic Press, New York, (1986)
10. A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*, Springer-Verlag Berlin Heidelberg, (2003)