

COOPERATIVE MOTION IN ONE DIMENSION

LOUIGI ADDARIO-BERRY, ERIN BECKMAN, AND JESSICA LIN

ABSTRACT. We prove distributional convergence for a family of random processes on \mathbb{Z} , which we call cooperative motions. The model generalizes the “totally asymmetric hipster random walk” introduced in [1]. We present a novel approach based on connecting a temporal recurrence relation satisfied by the cumulative distribution functions of the process to the theory of finite difference schemes for Hamilton-Jacobi equations [8]. We also point out some surprising lattice effects that can persist in the distributional limit, and propose several generalizations and directions for future research.

1. INTRODUCTION

1.1. Description of the model and the main result. Let $(D_n, n \geq 0)$ be a collection of independent, identically distributed integer random variables with common law ν . Fix a probability distribution μ on $\mathbb{Z} \cup \{-\infty, \infty\}$, and define a sequence $(X_n, n \geq 0)$ of extended real random variables as follows. Let X_0 be chosen according to μ . For $n \geq 0$, let $(\tilde{X}_n^i, 1 \leq i \leq m)$ be independent copies of X_n , and set

$$X_{n+1} = \begin{cases} X_n + D_n & \text{if } X_n = \tilde{X}_n^i \text{ for all } i = 1 \dots m, \\ X_n & \text{if } X_n \neq \tilde{X}_n^i \text{ for some } i, \end{cases} \quad (1.1)$$

where we use the convention that $\infty + r = \infty$ and $-\infty + r = -\infty$ for $r \in \mathbb{Z}$. We refer to the resulting sequence of random variables $(X_n, n \geq 0)$ as a cooperative motion with initial distribution μ and step size distribution ν .

We will principally consider the case when the steps $(D_n, n \geq 0)$ are Bernoulli(q)-distributed; in this case, we write $\text{CM}(m, q, \mu)$ for the law of the process $(X_n, n \geq 0)$ when started from initial distribution μ . Our main result is to show that $\text{CM}(m, q, \mu)$ processes are asymptotically Beta-distributed whenever the initial distribution μ is supported by \mathbb{Z} . This generalizes a result from [1], which is the case $m = 1$ of the next theorem.

Theorem 1.1. *Fix an integer $m \geq 1$, $q \in (0, 1)$ and any probability distribution μ on \mathbb{Z} , and let $(X_n, n \geq 0)$ be $\text{CM}(m, q, \mu)$ -distributed. Then*

$$\frac{1}{m+1} \left(\frac{m^m}{q} \right)^{\frac{1}{m+1}} \cdot \frac{X_n}{n^{1/(m+1)}} \xrightarrow{d} B, \quad (1.2)$$

where B is Beta($\frac{m+1}{m}, 1$)-distributed.

The $\text{CM}(m, q, \mu)$ processes are a type of random walk with delay. However, the amount of the delay is tied to the law of the process itself, since if X_n finds itself in an unlikely location, then the odds that $\tilde{X}_n^1, \dots, \tilde{X}_n^m$ are all equal to X_n are low. As such, the position and the rate of motion are highly dependent upon each other, which is the primary challenge in analyzing the process.

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1.2. Proof technique. Let B be Beta $\left(\frac{m+1}{m}, 1\right)$ -distributed. Our approach to establishing (1.2) is to work directly with the cumulative distribution function (CDF) of the rescaled random variable. In particular, we show that as $n \rightarrow \infty$, the CDF of $n^{-1/(m+1)}X_n$ converges to the CDF of $(m+1)\left(\frac{q}{m^m}\right)^{1/m+1}B$, which is

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \frac{m}{q^{\frac{1}{m}}(m+1)^{\frac{m+1}{m}}} x^{\frac{m+1}{m}} & \text{if } 0 \leq x \leq (m+1)\left(\frac{q}{m^m}\right)^{1/m+1}, \\ 1 & \text{otherwise.} \end{cases} \quad (1.3)$$

Our approach is based on the observation that the CDF of X_n in fact satisfies a finite-difference equation which approximates a first-order Hamilton-Jacobi equation. Note that if $F_k^n := \mathbf{P}(X_n < k)$, then since the steps D_n are $\{0, 1\}$ -valued, we have

$$\begin{aligned} F_k^{n+1} &= \mathbf{P}(X_n < k-1) + \mathbf{P}(X_n = k-1, X_{n+1} = k-1) \\ &= F_{k-1}^n + \mathbf{P}(X_n = k-1) - \mathbf{P}(X_n = k-1, X_{n+1} \neq k-1) \\ &= F_{k-1}^n + (F_k^n - F_{k-1}^n) - \mathbf{P}(X_n = k-1)\mathbf{P}(D_n = 1) \prod_{i=1}^m \mathbf{P}(\tilde{X}_n^i = k-1) \\ &= F_k^n - q(F_k^n - F_{k-1}^n)^{m+1}, \end{aligned} \quad (1.4)$$

and we may rewrite the final identity as

$$\begin{aligned} F_k^{n+1} - F_k^n &= -q(F_k^n - F_{k-1}^n)^{m+1} \\ &= -q|F_k^n - F_{k-1}^n|^{m+1}. \end{aligned} \quad (1.5)$$

The introduction of $|\cdot|$ in (1.5) is allowed since $F_k^n \geq F_{k-1}^n$. We write the recursion in the form of (1.5) because this makes (1.5) a discrete analogue (or finite difference scheme) of the first-order partial differential equation (PDE)

$$u_t + q|u_x|^{m+1} = 0 \quad \text{in } \mathbb{R} \times (0, \infty). \quad (1.6)$$

In a nutshell, our approach to proving Theorem 1.1 is to exploit this connection, showing that solutions of (1.5) closely approximate solutions of (1.6) after an appropriate rescaling, when n is large. The remainder of the introduction is principally dedicated to elaborating on the details of this approach and the challenges to carrying it out.

Equation (1.6) is an example of a nonlinear Hamilton-Jacobi equation of the form

$$u_t + H(u_x) = 0 \quad \text{in } \mathbb{R} \times (0, \infty),$$

with the Hamiltonian $H: \mathbb{R} \rightarrow \mathbb{R}$ defined by $H(p) = q|p|^{m+1}$. For general initial data, (1.6) fails to have classical, smooth solutions for all time. The theory of viscosity solutions introduced by Crandall and Lions [6, 7], which are continuous but need not be differentiable, provides a notion of weak solution for such equations. We will hereafter refer to Crandall-Lions viscosity solutions simply as continuous viscosity solutions. We provide an overview of relevant properties of viscosity solutions for Hamilton-Jacobi equations in Appendix A.

While continuous viscosity solutions are perhaps the most well-studied notion of weak solution for PDEs such as (1.6), our goal is to find a function $u(x, t)$ solving (1.6), which is meant to be an $n \rightarrow \infty$ analogue of the distribution function

$$\mathbf{P}\left(\frac{X_{[tn]}}{n^{1/(m+1)}} < x\right).$$

We note that for any initial distribution μ with $\mu(\mathbb{Z}) = 1$, we have

$$\mathbf{P}\left(\frac{X_0}{n^{1/(m+1)}} < x\right) \rightarrow \begin{cases} 1 & x > 0 \\ 0 & x < 0, \end{cases} \quad (1.7)$$

as $n \rightarrow \infty$, with the behaviour at $x = 0$ depending on the distribution μ . This implies that the continuous analogue $u(x, 0)$ we seek will necessarily have a discontinuity at $x = 0$. Such a discontinuity in the initial condition puts us outside of the framework of continuous viscosity solutions.

There have been several attempts to define an appropriate notion of discontinuous viscosity solutions (see [5] for some references). One notion, introduced by Barron and Jensen [4], is defined for *convex* Hamilton-Jacobi equations. This is our situation; the Hamiltonian $H(p) = q|p|^{m+1}$ in (1.6) is a convex function. (It is for this reason that we introduced absolute values in (1.5).) The Barron-Jensen theory applies exclusively to lower semicontinuous functions, which is why we choose to define $F_k^n = \mathbf{P}(X_n < k)$, instead of the more traditional definition of a CDF given by $\mathbf{P}(X_n \leq k)$. Of course, this makes practically no difference to the probabilistic analysis. We will refer to Barron-Jensen viscosity solutions as lsc (lower semicontinuous) viscosity solutions (see the Appendix for more details about the properties of these solutions which we make use of). Throughout this paper, every continuous (resp. lsc) viscosity solution we consider is in fact the unique continuous (resp. lsc) solution satisfying the PDE in question (see Theorem A.2 and Theorem A.7). Moreover, the two notions coincide for continuous functions. In particular, any lsc viscosity solution which is a continuous function is also a continuous viscosity solution (see Theorem A.6).

It turns out that the function F introduced in (1.3) is nothing more than $F(x) = u(x, 1)$, where $u(x, t)$ is the lsc viscosity solution of the initial value problem

$$\begin{cases} u_t + q|u_x|^{m+1} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = \mathbf{1}_{\{x>0\}} & \text{in } \mathbb{R}. \end{cases} \quad (1.8)$$

The lsc viscosity solution of (1.8) can be explicitly computed. In fact, for future use, we will compute the lsc viscosity solution of the more general PDE

$$\begin{cases} u_t^{a,b} + q|u_x^{a,b}|^{m+1} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u^{a,b}(x, 0) = a\mathbf{1}_{\{x \leq 0\}} + b\mathbf{1}_{\{x > 0\}} & \text{in } \mathbb{R}, \end{cases} \quad (1.9)$$

for $0 \leq a < b \leq 1$. Since (1.9) is a convex Hamilton-Jacobi equation, Theorem A.7 in the Appendix guarantees that the corresponding lsc viscosity solution is given by the Hopf-Lax formula from control theory,

$$u^{a,b}(x, t) = \inf_{y \in \mathbb{R}} \left\{ u^{a,b}(y, 0) + tH^*\left(\frac{x-y}{t}\right) \right\}, \quad (1.10)$$

where H^* is the Legendre transform of H , defined by

$$H^*(p) = \sup_{\alpha \in \mathbb{R}} (\alpha p - H(\alpha)).$$

For the Hamiltonian $H(p) = q|p|^{m+1}$, as H is superlinear ($\lim_{p \rightarrow \infty} \frac{H(p)}{|p|} = +\infty$) and $u^{a,b}(x, 0)$ is lower semicontinuous, the infimum in (1.10) is achieved. We may thus

compute explicitly that for this Hamiltonian,

$$\begin{aligned}
H^*(p) &= \sup_{\alpha \in \mathbb{R}} (\alpha p - q|\alpha|^{m+1}) \\
&= \frac{|p|^{\frac{m+1}{m}}}{(q(m+1))^{\frac{1}{m}}} - q^{-\frac{1}{m}} \left| \frac{p}{m+1} \right|^{\frac{m+1}{m}} \\
&= q^{-\frac{1}{m}} |p|^{\frac{m+1}{m}} \frac{m}{(m+1)^{\frac{m+1}{m}}}.
\end{aligned} \tag{1.11}$$

It follows that the lsc viscosity solution $u^{a,b}$ of (1.9) is given by

$$\begin{aligned}
u^{a,b}(x,t) &= \inf_{y \in \mathbb{R}} \left\{ a \mathbb{1}_{\{y \leq 0\}} + b \mathbb{1}_{\{y > 0\}} + t H^* \left(\frac{x-y}{t} \right) \right\} \\
&= \inf_{y \in \mathbb{R}} \left\{ a \mathbb{1}_{\{y \leq 0\}} + b \mathbb{1}_{\{y > 0\}} + t q^{-\frac{1}{m}} \left| \frac{x-y}{t} \right|^{\frac{m+1}{m}} \frac{m}{(m+1)^{\frac{m+1}{m}}} \right\} \\
&= \inf_{y \in \mathbb{R}} \left\{ a \mathbb{1}_{\{y \leq 0\}} + b \mathbb{1}_{\{y > 0\}} + \frac{1}{t^{\frac{1}{m}}} q^{-\frac{1}{m}} \frac{m}{(m+1)^{\frac{m+1}{m}}} |x-y|^{\frac{m+1}{m}} \right\}.
\end{aligned}$$

A straightforward analysis yields that the preceding infimum is achieved at

$$y = \begin{cases} 0 & \text{if } 0 \leq \left(\frac{x^{m+1}}{t} \right)^{\frac{1}{m}} \leq (b-a) q^{\frac{1}{m}} \frac{(m+1)^{\frac{m+1}{m}}}{m}, \\ x & \text{otherwise.} \end{cases}$$

This implies that

$$u^{a,b}(x,t) = \begin{cases} a & \text{if } x \leq 0, \\ a + \frac{m}{q^{\frac{1}{m}}(m+1)^{\frac{m+1}{m}}} \left(\frac{x^{m+1}}{t} \right)^{\frac{1}{m}} & \text{if } 0 \leq \left(\frac{x^{m+1}}{t} \right)^{\frac{1}{m}} \leq (b-a) q^{\frac{1}{m}} \frac{(m+1)^{\frac{m+1}{m}}}{m}, \\ b & \text{otherwise.} \end{cases} \tag{1.12}$$

In the case when $a = 0$, and $b = 1$ (so for u solving (1.8)), we may rewrite this as

$$u(x,t) = \begin{cases} 0 & \text{if } x \leq 0, \\ \frac{m}{q^{\frac{1}{m}}(m+1)^{\frac{m+1}{m}}} \left(\frac{x^{m+1}}{t} \right)^{\frac{1}{m}} & \text{if } 0 \leq x \leq (m+1) \left(\frac{qt}{m^m} \right)^{1/m+1}, \\ 1 & \text{otherwise,} \end{cases} \tag{1.13}$$

which agrees with the rescaled Beta CDF given in (1.3) when $t = 1$. With regards to demonstrating the convergence of the finite difference scheme (1.5) to solutions of (1.6), we begin by recalling a robust result of Crandall and Lions [8]. In [8], the authors identify sufficient conditions for functions defined by finite difference schemes on a space time mesh $\Delta_x \mathbb{Z} \times \Delta_t \mathbb{N}$ to converge to the continuous viscosity solution of the corresponding Hamilton-Jacobi equation (such as (1.6)). Their general result is stated as Theorem 2.3, below. Upon an appropriate scaling, we may convert (1.5) to a finite difference relation on $\Delta_x \mathbb{Z} \times \Delta_t \mathbb{N}$. Theorem 2.3 implies that, if (1.5) satisfies a monotonicity condition (see Definition 2.2) and $F_k^0 := u_0(k\Delta_x)$ is the discretization of a Lipschitz continuous function u_0 on the mesh $\Delta_x \mathbb{Z}$, then for all sufficiently small Δ_x , the values F_k^N defined by the finite difference scheme are uniformly close to solutions $u(k\Delta_x, N\Delta_t)$ of the PDE with $u(x,0) = u_0(x)$, for $N\Delta_t$ lying in any compact time interval $[0, T]$. The Crandall–Lions theory, however, relies upon the initial data being Lipschitz continuous, as well as using the theory of continuous viscosity solutions. Since we aim to show that the CDFs of the rescaled $\text{CM}(m, q, \mu)$ random variables

$(X_n n^{-1/(m+1)}, n \geq 0)$ converge to the lsc viscosity solution of (1.8), this precludes a direct application of the results of [8] to prove Theorem 1.1. Furthermore, to the best of our knowledge, no numerical approximation results analogous to those of [8] have been proved for lsc viscosity solutions.

Probabilistically, the Lipschitz continuity required by the Crandall-Lions theory is also an issue: it means that the CDF of $X_0/n^{1/(m+1)}$ should be a discretization of a Lipschitz function, with Lipschitz constant independent of n ; but for a fixed initial distribution, this is impossible (recall (1.7)). To obtain such Lipschitz continuity, the Crandall-Lions theory thus requires the initial condition for the discrete process to depend on the mesh size, which probabilistically translates to requiring the initial distribution of the cooperative motion to depend on the target time n at which we wish to observe the process.

In order to make use of the results of [8] in our setting, we use further properties of the probabilistic model in order to demonstrate convergence to the lsc viscosity solution (which corresponds to the Beta-distributed limit in Theorem 1.1). In particular, we prove a discrete stochastic monotonicity result, Lemma 3.2, which allows us to couple the process started from different initial distributions. This coupling is surprisingly delicate; it is not the case that the cooperative motion evolution preserves stochastic ordering for arbitrary initial distributions. However, we prove that it preserves stochastic ordering whenever the initial distribution is not too singular (i.e. when all atoms satisfy a quantitative upper bound, depending on q and m); see Proposition 3.1. Having established this allows us to use the results of Crandall and Lions [8] to prove convergence to an lsc viscosity solution. We can then stochastically sandwich the evolution started from any initial conditions using Lipschitz-continuous (n -dependent) initial conditions, up to an error term which can be made arbitrarily small (after rescaling by $n^{1/(m+1)}$). This allows us to demonstrate the convergence in (1.2) for sufficiently non-singular initial distributions. We then conclude by showing that any initial distribution “relaxes” to a sufficiently non-singular distribution in a bounded number of steps.

We mention that a recurrence similar to (1.5) can be written for the probability mass function $p_k^n = \mathbf{P}(X_n = k)$:

$$p_k^{n+1} - p_k^n = -q \left((p_k^n)^{m+1} - (p_{k-1}^n)^{m+1} \right). \quad (1.14)$$

This recurrence can be interpreted as a discretization of the scalar conservation law,

$$v_t = -q(v^{m+1})_x. \quad (1.15)$$

Indeed, this connection was observed in [1] in the special case when $m = 1$, and the proof in [1] of the $m = 1$ case of Theorem 1.1 relied upon similar numerics/PDE L^1 convergence results for finite difference schemes of scalar conservation laws. In particular, rescaled solutions of (1.14) converge in L^1 to the unique entropy solutions of (1.15). From the theory of PDEs, it is well-known that in the one-dimensional setting, entropy solutions of (1.15) correspond precisely to derivatives of viscosity solutions of (1.6). This motivated our approach of working directly with the CDFs in this paper, and using viscosity solutions methods in this setting. The advantages of working with viscosity solutions include (a) the fact that the solution theory, at least as it relates to such probabilistic models, is better developed for viscosity solutions than for the corresponding entropy solutions, and (b) the fact that working in the “integrated” setting gives the solutions greater regularity, which makes the resulting proofs more direct.

The rest of the paper proceeds as follows. In Section 2, we review the results of Crandall and Lions [8] and use them to demonstrate convergence of CDFs of a

rescaled $\text{CM}(m, q, \mu)$ process with a “diffuse” initial condition, which approximates a Lipschitz continuous function. In Section 3, we show convergence of CDFs of a rescaled $\text{CM}(m, q, \mu)$ process with initial distribution μ which has no overly large atoms in its support. In Section 4, we remove this hypothesis on the size of the atoms of μ , and complete the proof of Theorem 1.1. Section 5 concerns the limitations of the approach taken in this paper, and includes Theorem 5.4, which presents a provable obstacle to applying our methodology to establish convergence of cooperative motion-type processes with non-Bernoulli step sizes. This section also presents Theorem 5.1, which shows that when the step size is an integer multiple of a Bernoulli, the resulting lattice effects lead to limits which are mixtures of Beta distributions. Finally, Appendix A provides an overview of continuous and lsc viscosity solutions, and describes several important properties of such solutions that we use throughout the paper.

1.3. Notation. Before proceeding, we introduce some terminological conventions. Given a random variable X , we define the CDF $F_X : \mathbb{R} \rightarrow [0, 1]$ of X by $F_X(x) = \mathbf{P}(X < x)$; As mentioned in the introduction, we use this definition rather than the standard $F_X(x) = \mathbf{P}(X \leq x)$ to make it easier to appeal to the relevant PDE theory, which has been developed for lower-semicontinuous functions.

We say a function $F : \mathbb{R} \rightarrow [0, 1]$ is a CDF if it is the CDF of an \mathbb{R} -valued random variable, and that F is an extended CDF if it is the CDF of an extended random variable (i.e. a random variable taking values in $\mathbb{R} \cup \{\pm\infty\}$).

For random variables X, Y taking values in $\mathbb{R} \cup \{\pm\infty\}$, we write $X \leq Y$ and say that Y stochastically dominates X if $\mathbf{P}(X < x) \geq \mathbf{P}(Y < x)$ for all $x \in \mathbb{R}$.

Let $(X_n, n \geq 0)$ and $(\tilde{X}_n, n \geq 0)$ be $\text{CM}(m, q, \mu)$ and $\text{CM}(m, q, \tilde{\mu})$ -distributed, respectively. Suppose that $\tilde{X}_0 \leq X_0$. Then we say that the CM-evolution is *stochastically monotone* for μ and $\tilde{\mu}$ if $\tilde{X}_n \leq X_n$ for all $n \geq 0$. In other words, the CM-evolution is stochastically monotone for μ and $\tilde{\mu}$ if it preserves their stochastic ordering in time.

2. FINITE DIFFERENCE SCHEMES FOR DIFFUSE INITIAL CONDITIONS

As mentioned in Section 1, our approach is to interpret CDFs of the discrete random variables $(X_n, n \geq 0)$ as solutions of a finite difference scheme. As before, fix $m \in \mathbb{N}$ with $m \geq 1$, $q \in (0, 1)$, and a probability distribution μ supported on $\mathbb{Z} \cup \{-\infty, \infty\}$. Let $(X_n, n \geq 0)$ be $\text{CM}(m, q, \mu)$ -distributed, and for $k \in \mathbb{Z}$ write $F_k^n = F_k^n(\mu) = \mathbf{P}(X_n < k) = \mu[-\infty, k)$. (We suppress the dependence on m and q as they are fixed throughout, and also suppress the dependence on μ whenever possible.) Then $(F_k^n)_{k \in \mathbb{Z}, n \in \mathbb{N}}$ is defined by

$$\begin{cases} F_k^{n+1} - F_k^n = -q (F_k^n - F_{k-1}^n)^{m+1} & n \geq 0, k \in \mathbb{Z}, \\ F_k^0 = \mu[-\infty, k) & k \in \mathbb{Z}. \end{cases} \quad (2.1)$$

Since F_k^n is nondecreasing in k for all $n \in \mathbb{N}$, (2.1) can be rewritten as

$$\begin{cases} F_k^{n+1} - F_k^n = -q |F_k^n - F_{k-1}^n|^{m+1} & n \geq 0, k \in \mathbb{Z}, \\ F_k^0 = \mu[-\infty, k) & k \in \mathbb{Z}, \end{cases} \quad (2.2)$$

and the function defined by (2.2) is identical to the function defined by (2.1). We will use (2.1) and (2.2) interchangeably, and will also use the fact that F_k^n is nondecreasing in k , for all $n \in \mathbb{N}$, frequently in what follows.

The main result of this section is the following proposition, which states that solutions of the recurrence relation from (2.2), with nondecreasing, Lipschitz initial data converge to solutions of the appropriate Hamilton-Jacobi equation.

Proposition 2.1. *Let u_0 be a Lipschitz-continuous extended CDF with Lipschitz constant K . Fix $N \in \mathbb{N}$ and define a probability distribution μ_N on $\mathbb{Z} \cup \{-\infty, \infty\}$ by*

$$\mu_N[-\infty, k] := u_0(kN^{-1/(m+1)})$$

for $k \in \mathbb{Z}$. Let $(X_n, n \geq 0)$ be $CM(m, q, \mu_N)$ -distributed, and let $F_k^n = F_k^n(\mu_N) = \mathbf{P}(X_n < k)$. Finally, fix $T > 0$. Then there exist $N_0 = N_0(q, m, K)$ and $c = c(K, m, q, T)$ such that if $N \geq N_0$,

$$\sup_{0 \leq t \leq T} \sup_{k \in \mathbb{Z}} \left| F_k^{\lfloor Nt \rfloor} - u\left(\frac{k}{N^{1/(m+1)}}, t\right) \right| \leq \frac{c}{N^{1/2}}, \quad (2.3)$$

where u is the continuous viscosity solution of

$$\begin{cases} u_t + q|u_x|^{m+1} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}. \end{cases} \quad (2.4)$$

It follows that u is an extended CDF and that

$$\sup_{x \in \mathbb{R}} \left| \mathbf{P}\left(\frac{X_N}{N^{1/(m+1)}} < x\right) - u(x, 1) \right| \leq \frac{c}{N^{1/2}}. \quad (2.5)$$

In order to prove this proposition, we require the framework of monotone finite difference schemes for Hamilton–Jacobi equations. We next introduce this framework, and relate it to the evolution of the CDF of cooperative motion.

We may imagine numerically approximating the solution of a Hamilton–Jacobi equation of the form

$$\begin{cases} u_t + H(u_x) = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R} \end{cases} \quad (2.6)$$

as follows. Fix temporal and spatial mesh sizes $(\Delta_t$ and Δ_x , respectively). Set $U_k^0 = u_0(k\Delta_x)$ for $k \in \mathbb{Z}$, and for $n \geq 0$ define U_k^{n+1} by

$$U_k^{n+1} = G(U_k^n, U_{k-1}^n) = U_k^n - \Delta_t H\left(\frac{U_k^n - U_{k-1}^n}{\Delta_x}\right), \quad (2.7)$$

so $G(y, z) = G^\Delta(y, z) = y - \Delta_t H\left(\frac{y-z}{\Delta_x}\right)$, where $\Delta = (\Delta_x, \Delta_t)$. We may use (2.7) to define a rescaled field of values

$$u^\Delta : \Delta_x \mathbb{Z} \times \Delta_t \mathbb{N} \rightarrow \mathbb{R}$$

by setting $u^\Delta(k\Delta_x, n\Delta_t) := U_k^n$. With this definition, (2.7) is equivalent to the statement that

$$\frac{u^\Delta(k\Delta_x, n\Delta_t + \Delta_t) - u^\Delta(k\Delta_x, n\Delta_t)}{\Delta_t} + H\left(\frac{u^\Delta(k\Delta_x, n\Delta_t) - u^\Delta(k\Delta_x - \Delta_x, n\Delta_t)}{\Delta_x}\right) = 0.$$

This indeed looks, formally, like a discretization of (2.6) on the space-time mesh $\Delta_x \mathbb{Z} \times \Delta_t \mathbb{N}$. We refer to (2.7) as a *finite difference scheme* for the initial value problem (2.6). It turns out that, under suitable regularity assumptions on the initial condition u_0 and the Hamiltonian H , the sufficient conditions on (2.7) for u^Δ , or equivalently $(U_k^n)_{k \in \mathbb{Z}, n \in \mathbb{N}}$, to well-approximate u as Δ_t and $\Delta_x \rightarrow 0$ are *consistency* and *monotonicity*. The consistency condition states that when G is written in differenced form, i.e.

$$G(U_k^n, U_{k-1}^n) = U_k^n - \Delta_t g\left(\frac{U_k^n - U_{k-1}^n}{\Delta_x}\right)$$

for some function $g : \mathbb{R} \rightarrow \mathbb{R}$, then $g(p) = H(p)$ for all $p \in \mathbb{R}$. This is trivially satisfied in our setting by the scheme defined in (2.7).

The more subtle condition is monotonicity, which we next define.

Definition 2.2. A scheme of the form (2.7) is monotone on $[\lambda, \Lambda] \subseteq \mathbb{R}$ if $G(U_k^n, U_{k-1}^n)$ is a nondecreasing function of each argument so long as

$$\lambda \leq (\Delta_x)^{-1} (U_k^n - U_{k-1}^n) \leq \Lambda. \quad (2.8)$$

We now state the main result from [8], specialized to the one-dimensional setting of the current paper, on the quality of approximation provided by monotone finite difference schemes for Hamilton-Jacobi equations.

Theorem 2.3. [Theorem 1, [8]] Let $u : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ be the continuous viscosity solution of

$$\begin{cases} u_t + H(u_x) = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}, \end{cases} \quad (2.9)$$

where $H : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and u_0 is bounded and Lipschitz continuous with Lipschitz constant K . Fix $\Delta_x > 0$ and $\Delta_t > 0$, let $U_k^0 := u_0(k\Delta_x)$, and define U_k^n by a general scheme of the form (2.7).

If (2.7) is consistent and monotone on $[-(K+1), K+1]$, then for any $T > 0$, there exists c , depending on $\sup|u_0|, K, H$, and T so that

$$\sup_{n \in \mathbb{N}, n\Delta_t \in [0, T]} \sup_{k \in \mathbb{Z}} |U_k^n - u(k\Delta_x, n\Delta_t)| \leq c\sqrt{\Delta_t}. \quad (2.10)$$

Before connecting Theorem 2.3 to cooperative motion, it is instructive to further discuss the meaning and value of monotonicity in this setting. (The following discussion is inspired by the proof of [8, Proposition 3.1].)

Fix $K > 0$ and two sets of initial conditions $(U_k^0)_{k \in \mathbb{Z}}$ and $(\tilde{U}_k^0)_{k \in \mathbb{Z}}$ with $U_k^0 \leq \tilde{U}_k^0$, then set $U_k^{n+1} = G(U_k^n, U_{k-1}^n)$ and $\tilde{U}_k^{n+1} = G(\tilde{U}_k^n, \tilde{U}_{k-1}^n)$ for $n \geq 0$ as in (2.7).

Suppose that G is monotone on $[-K, K]$, and that

$$\frac{|U_k^0 - U_{k-1}^0|}{\Delta_x} \leq K \quad \text{and} \quad \frac{|\tilde{U}_k^0 - \tilde{U}_{k-1}^0|}{\Delta_x} \leq K \quad (2.11)$$

for all k . Then monotonicity implies that

$$U_k^1 = G(U_k^0, U_{k-1}^0) \leq G(\tilde{U}_k^0, \tilde{U}_{k-1}^0) = \tilde{U}_k^1. \quad (2.12)$$

Next, let $(W_k^0)_{k \in \mathbb{Z}}$ be any initial condition with $\sup_k \Delta_x^{-1} |W_k^0 - W_{k-1}^0| \leq K$, and set $W_k^{n+1} = G(W_k^n, W_{k-1}^n)$ for $n \geq 0$ and $k \in \mathbb{Z}$. Write $\lambda = \sup_k |W_k^0 - U_k^0|$. Let $V_k^0 = U_k^0 + \lambda$, and set $V_k^1 = G(V_k^0, V_{k-1}^0) = U_k^1 + \lambda$. By the choice of λ , we have $W_k^0 \leq V_k^0$. Then monotonicity gives that

$$W_k^1 \leq V_k^1 = U_k^1 + \lambda,$$

and a symmetric argument gives that $W_k^1 \geq U_k^1 - \lambda$, so

$$\sup_k |W_k^1 - U_k^1| \leq \lambda.$$

We apply this with the specific choice of initial condition $W_k^0 = U_{k-1}^0$. Since $W_k^1 = U_{k-1}^1$, the preceding bound gives

$$\sup_k |U_k^1 - U_{k-1}^1| = \sup_k |U_k^1 - W_k^1| \leq \lambda = \sup_k |U_k^0 - W_k^0| = \sup_k |U_k^0 - U_{k-1}^0| \leq K\Delta_x.$$

A similar analysis allows us to conclude that

$$\sup_k |\tilde{U}_k^1 - \tilde{U}_{k-1}^1| \leq K\Delta_x.$$

By the two preceding bounds and (2.12), it follows by induction that $U_k^n \leq \tilde{U}_k^n$ for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ and that $\sup_k |U_k^n - U_{k-1}^n| \leq K\Delta_x$ for all n . In short, equation (2.8),

which can be viewed as a type of discrete Lipschitz bound on U_k^n , allows one to show that an order relation between two initial conditions persists for all positive times.

Remark 2.4. Whenever the initial condition $(U_k^0)_{k \in \mathbb{Z}}$ is nondecreasing in k , the above argument shows that if G is monotone on $[0, K]$ and $\sup_k (U_k^0 - U_{k-1}^0) \leq K \Delta_x$, then $(U_k^n)_{k \in \mathbb{Z}}$ is non-decreasing in k and $\sup_k (U_k^n - U_{k-1}^n) \leq K \Delta_x$ for all $n \in \mathbb{N}$. It follows from this that if u_0 is non-decreasing, then in order to verify the condition of Theorem 2.3 one need only check that (2.7) is monotone on $[0, K + 1]$.

We now specialize the above discussion to the specific setting of our paper, so again let $F_k^n = \mathbf{P}(X_n < k)$ where $(X_n, n \geq 0)$ is $\text{CM}(m, q, \mu)$ -distributed. Given spatial and temporal mesh sizes $(\Delta_x$ and Δ_t , respectively), we may use the field of values $(F_k^n)_{k \in \mathbb{Z}, n \in \mathbb{N}}$ to define a rescaled field $f = f_{\Delta_t, \Delta_x} : \Delta_x \mathbb{Z} \times \Delta_t \mathbb{N} \rightarrow \mathbb{R}$ by setting $f(k\Delta_x, n\Delta_t) := F_k^n$.

In order to identify an appropriate scaling relationship between Δ_x and Δ_t , we seek a continuous space-time scaling which preserves the PDE. In particular, if u solves (1.8), then for any $\rho \in \mathbb{R}$, $u_\rho(x, t) := u(\rho x, \rho^{m+1}t)$ also solves (1.8). This suggests that the temporal and spatial mesh sizes should satisfy the relation

$$(\Delta_x)^{m+1} = \Delta_t. \quad (2.13)$$

With this scaling relation, we may rewrite (2.1) as

$$F_k^{n+1} = F_k^n - q\Delta_t \left(\frac{F_k^n - F_{k-1}^n}{\Delta_x} \right)^{m+1}, \quad (2.14)$$

which, since F_k^n is non-decreasing in k , we may re-express as

$$F_k^{n+1} = F_k^n - q\Delta_t \left| \frac{F_k^n - F_{k-1}^n}{\Delta_x} \right|^{m+1}. \quad (2.15)$$

This equation has precisely the form of (2.7) with $G(y, z) = y - q\Delta_t | \frac{y-z}{\Delta_x} |^{m+1} = y - q|y-z|^{m+1}$, the second equality holding due to (2.13). The fact that in this setting G does not in fact depend on Δ_x and Δ_t means that to verify monotonicity, one may assume that Δ_x takes any fixed positive value – say $\Delta_x = 1$.

Now fix probability distributions $\mu, \tilde{\mu}$ on $\mathbb{Z} \cup \{\pm\infty\}$ with $\tilde{\mu} \leq \mu$, let $(X_n, n \geq 0)$ and $(\tilde{X}_n, n \geq 0)$ be $\text{CM}(m, q, \mu)$ and $\text{CM}(m, q, \tilde{\mu})$ -distributed, respectively, and set $F_k^n = \mathbf{P}(X_n < k)$ and $\tilde{F}_k^n = \mathbf{P}(\tilde{X}_n < k)$, so $(F_k^n)_{k \in \mathbb{Z}, n \in \mathbb{N}}$ and $(\tilde{F}_k^n)_{k \in \mathbb{Z}, n \in \mathbb{N}}$ both satisfy (2.15) but with different initial conditions. The fact that $\tilde{\mu} \leq \mu$ means that $F_k^0 \leq \tilde{F}_k^0$.

For a given $\Lambda > 0$, if G is monotone on $[0, \Lambda]$ and $\sup_{k \in \mathbb{Z}} (F_k^0 - F_{k-1}^0) \leq \Lambda$ and $\sup_{k \in \mathbb{Z}} (\tilde{F}_k^0 - \tilde{F}_{k-1}^0) \leq \Lambda$, then (2.12) gives that $F_k^1 \leq \tilde{F}_k^1$, and inductively that $F_k^n \leq \tilde{F}_k^n$ for all n . In other words, we can think of monotonicity as a sufficient condition which guarantees that the cooperative motion will preserve stochastic ordering in time. In Section 3, we will use a variation of this approach to identify the value of Λ , and thereby a sufficient condition, which guarantees stochastic monotonicity.

Proof of Proposition 2.1. Let $N_0 := ([q(m+1)]^{1/m}(K+1))^{(m+1)}$ and fix $N \geq N_0$. We choose $\Delta_x = N^{-1/(m+1)}$, and $\Delta_t = N^{-1}$, so that Δ_x and Δ_t satisfy (2.13). This implies that $F_k^{(\cdot)}(\mu_N)$ is defined by (2.15). The proof relies upon verifying the hypotheses of Theorem 2.3 for $U_k^n = F_k^n$. As u_0 is nondecreasing, by Remark 2.4, we only need to verify that (2.15) or, equivalently, (2.14) is monotone in $[0, K + 1]$.

To verify monotonicity of (2.14), we differentiate

$$G(F_k^n, F_{k-1}^n) = F_k^n - q\Delta_t \left(\frac{F_k^n - F_{k-1}^n}{\Delta_x} \right)^{m+1}$$

in each argument, in the region $0 \leq (\Delta_x)^{-1}(F_k^n - F_{k-1}^n) \leq K + 1$. Differentiating with respect to F_k^n , we have

$$\begin{aligned} 1 - q(m+1) \frac{\Delta_t}{\Delta_x} \left(\frac{F_k^n - F_{k-1}^n}{\Delta_x} \right)^m &\geq 1 - q(m+1) \frac{\Delta_t}{\Delta_x} (K+1)^m \\ &= 1 - q(m+1) (\Delta_x)^m (K+1)^m. \end{aligned}$$

As $N \geq N_0$, we have that

$$(\Delta_x)^m = N^{-m/(m+1)} \leq [q(m+1)]^{-1} (K+1)^{-m},$$

which implies that $G(\cdot, F_{k-1}^n)$ is nondecreasing. It similarly follows that $G(F_k^n, \cdot)$ is nondecreasing. This implies that (2.14) is monotone on $[0, K+1]$, so by Theorem 2.3, we then have that for $N \geq N_0$, for any $T > 0$, there is $c = c(K, m, q, T)$ such that

$$\sup_{0 \leq j/N \leq T} \sup_{k \in \mathbb{Z}} \left| F_k^j - u \left(\frac{k}{N^{1/(m+1)}}, \frac{j}{N} \right) \right| \leq cN^{-\frac{1}{2}};$$

recall that $\Delta_t = N^{-1}$, so $j/N = j\Delta_t$. We may rewrite this bound as

$$\sup_{0 \leq t \leq T} \sup_{k \in \mathbb{Z}} \left| F_k^{\lfloor Nt \rfloor} - u \left(\frac{k}{N^{1/(m+1)}}, \frac{1}{N} \lfloor Nt \rfloor \right) \right| \leq cN^{-\frac{1}{2}}.$$

By Proposition A.4, the continuous viscosity solution u solving (2.4) is globally Lipschitz continuous in space and time. Therefore,

$$\begin{aligned} &\sup_{0 \leq t \leq T} \sup_{k \in \mathbb{Z}} \left| F_k^{\lfloor Nt \rfloor} - u \left(\frac{k}{N^{1/(m+1)}}, t \right) \right| \\ &\leq \sup_{0 \leq t \leq T} \sup_{k \in \mathbb{Z}} \left| F_k^{\lfloor Nt \rfloor} - u \left(\frac{k}{N^{1/(m+1)}}, \frac{1}{N} \lfloor Nt \rfloor \right) \right| \\ &\quad + \sup_{0 \leq t \leq T} \sup_{k \in \mathbb{Z}} \left| u \left(\frac{k}{N^{1/(m+1)}}, \frac{1}{N} \lfloor Nt \rfloor \right) - u \left(\frac{k}{N^{1/(m+1)}}, t \right) \right| \\ &\leq cN^{-\frac{1}{2}} + C \sup_{0 \leq t \leq T} \left| \frac{1}{N} \lfloor Nt \rfloor - t \right| \leq \tilde{c}N^{-\frac{1}{2}}, \end{aligned}$$

and this yields (2.3); equation (2.5) follows as it is simply a restating of (2.3) in the special case when $t = 1$. Finally, (2.5) gives that $u(x, 1)$ is the pointwise limit of a CDF, so is itself an extended CDF. \square

3. “GOOD” SINGULAR INITIAL CONDITIONS

The convergence results of the previous section require that the finite difference scheme $(F_k^n)_{k \in \mathbb{Z}, n \in \mathbb{N}}$ begins with an initial condition μ_N which is a discretization of a Lipschitz function at scale Δ_x (depending on N). In this section, we build on those convergence results to prove distributional limit theorems for certain fixed (rather than varying in N) initial conditions. Let

$$p^* = \left(\frac{1}{q(m+1)} \right)^{1/m}, \quad (3.1)$$

and note that $p^* > 1/2$ since $(m+1)^{-1/m} \geq 1/2$ for all $m \geq 1$ and $q \in (0, 1)$. We say that an extended probability distribution μ is p^* -bounded if

$$\sup_{x \in \mathbb{Z} \cup \{-\infty, \infty\}} \mu(\{x\}) < p^*.$$

The goal of this section is to prove the following proposition, which essentially states that Theorem 1.1 holds for p^* -bounded initial conditions.

Proposition 3.1. *Let $(X_n, n \geq 0)$ be $\text{CM}(m, q, \mu)$ -distributed with μ a probability distribution on \mathbb{Z} . If μ is p^* -bounded, then*

$$\lim_{n \rightarrow \infty} \mathbf{P} \left(\frac{X_n}{n^{1/(m+1)}} < x \right) = u(x, 1)$$

uniformly in x , where $u(x, t)$ is given by (1.13).

The proof of Proposition 3.1 relies on comparison between the CM-evolution with p^* -bounded initial conditions to CM-evolutions with Lipschitz continuous initial conditions. To establish the possibility of such comparisons, we prove that the CM evolution is stochastically monotone on a much broader class of initial conditions than what is covered by Proposition 2.1. (It may be useful to revisit the discussion preceding the proof of Proposition 2.1 at this point.) We first show for the class of p^* -bounded distributions, stochastic ordering is preserved by one time-step of the CM evolution. We then show that the CDFs at future time steps remain in the family of p^* -bounded distributions. This is exactly the content of the next two lemmas:

Lemma 3.2. *Let μ_X and μ_Y be p^* -bounded probability distributions on $\mathbb{Z} \cup \{-\infty, \infty\}$, and let $(X_n, n \geq 0)$ be $\text{CM}(m, q, \mu_X)$ -distributed and $(Y_n, n \geq 0)$ be $\text{CM}(m, q, \mu_Y)$ -distributed. If $Y_0 \leq X_0$ then $Y_1 \leq X_1$.*

Lemma 3.3. *Let $(X_n, n \geq 0)$ be $\text{CM}(m, q, \mu)$ -distributed and define $\mathbf{P}(X_n = k) = p_k^n$. Then for all $j \in \mathbb{Z}$,*

$$p_j^{n+1} \leq \max(p_{j-1}^n, p_j^n), \quad (3.2)$$

where there is equality if and only if $p_{j-1}^n = p_j^n$.

The requirement that μ_X and μ_Y are p^* -bounded in Lemma 3.2 is necessary. To see this, fix non-negative integer random variables $X_0 \leq Y_0$ with $\mathbf{P}(Y_0 = 0) = \tilde{p} < p = \mathbf{P}(X_0 = 0)$. Then by (1.14),

$$\begin{aligned} \mathbf{P}(X_1 < 1) &= \mathbf{P}(X_1 = 0) = p - qp^{m+1}, \\ \mathbf{P}(Y_1 < 1) &= \mathbf{P}(Y_1 = 0) = \tilde{p} - q\tilde{p}^{m+1}. \end{aligned}$$

In order to have $X_1 \leq Y_1$, we thus require that $\tilde{p} - q\tilde{p}^{m+1} \leq p - qp^{m+1}$, or in other words that $p - qp^{m+1}$ is non-decreasing in p . Differentiating, we see that this is equivalent to requiring that

$$1 - (m+1)qp^m \geq 0,$$

which is true precisely when $p \leq p^*$. (For later use, write $f(p) := p - qp^{m+1}$; we will use that f is increasing on $[0, p^*]$ and decreasing on $[p^*, 1]$.)

Proof of Lemma 3.2. Let $G(y, z) = y - q|y - z|^{m+1}$. Note that for $0 \leq z < y$, $G(y, z)$ is increasing in z , and for $0 \leq z \leq y$, G is increasing in y provided that

$$1 - (m+1)q(y-z)^m \geq 0,$$

or in other words provided that $y - z \leq p^*$. Therefore, G is monotone on $[0, p^*]$.

Now write $F_k^0 = \mathbf{P}(X_0 < k) = \mu_X[-\infty, k)$ and $\tilde{F}_k^0 = \mathbf{P}(Y_0 < k) = \mu_Y[-\infty, k)$. Since $Y_0 \leq X_0$, we have that for all $k \in \mathbb{Z}$, $F_k^0 \leq \tilde{F}_k^0$, and moreover, $F_k^0 - F_{k-1}^0 \leq p^*$ and $\tilde{F}_k^0 - \tilde{F}_{k-1}^0 \leq p^*$. Since G is monotone on $[0, p^*]$ it follows that

$$\mathbf{P}(X_1 < k) = F_k^1 = G(F_k^0, F_{k-1}^0) \leq G(\tilde{F}_k^0, \tilde{F}_{k-1}^0) = \tilde{F}_k^1 = \mathbf{P}(Y_1 < k),$$

so $Y_1 \leq X_1$, as required. \square

We next introduce an additional technical lemma needed in the proof of Lemma 3.3.

Lemma 3.4. Let $g(x, y) = f(x) - f(y)$ with $f(x) = x - qx^{m+1}$. Then $g(a, b) > 0$ whenever $a > b \geq 0$ and $a + b \leq 1$. Under the additional constraint $a > p^*$, we have

$$g(a, b) > \min(g(p^*, 1 - p^*), 1 - q) > 0. \quad (3.3)$$

Proof. First, we note that

$$\begin{aligned} \frac{\partial}{\partial x} g(x, y) &= 1 - q(m+1)x^m, \\ \frac{\partial}{\partial y} g(x, y) &= -1 + q(m+1)y^m. \end{aligned}$$

We are concerned with the behavior of the function g in the regions A and B shown in Figure 1. Formally, if $C = \{(x, y) : 0 \leq y < x, y \leq 1 - x\}$, then $A = \{(x, y) \in C : x \leq p^*\}$ and $B = \{(x, y) \in C : x > p^*\}$.

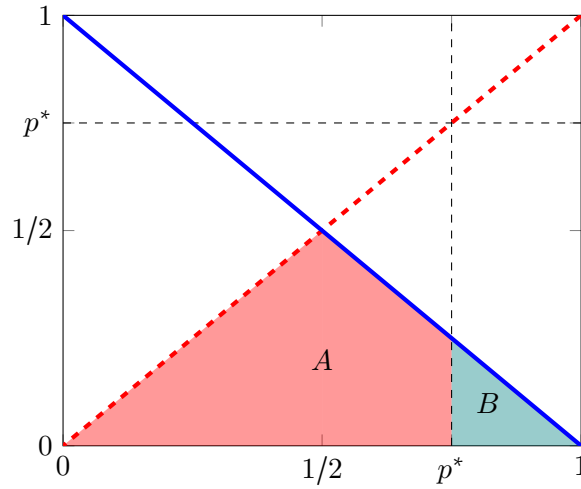


FIGURE 1. Lemma 3.4 states that $g(a, b)$ is positive for $(a, b) \in A$ and is greater than $\min(g(p^*, 1 - p^*), 1 - q)$ for $(a, b) \in B$.

If $(x, y) \in A$ then since f is increasing on $[0, p^*]$ and in this region $0 \leq y < x \leq p^*$, it follows that $g(x, y) = f(x) - f(y) > 0$.

To determine the behavior in region B , notice that in this region, $\frac{\partial g}{\partial x} < 0$ and $\frac{\partial g}{\partial y} < 0$. Therefore,

$$\inf_{(x, y) \in B} g(x, y) = \inf_{x \in [p^*, 1]} g(x, 1 - x)$$

Moreover,

$$\frac{\partial^2}{\partial x^2} g(x, 1 - x) = q(m+1)m((1-x)^{m-1} - x^{m-1});$$

for $x > 1/2$, the difference $(1-x)^{m-1} - x^{m-1}$ is strictly negative, so $g(x, 1-x)$ is strictly concave for $x \in (1/2, 1)$. Since $p^* > 1/2$, we thus have

$$\inf_{(x, y) \in B} g(x, y) = \inf_{x \in [p^*, 1]} g(x, 1 - x) > \inf_{x \in \{1/2, 1\}} g(x, 1 - x). \quad (3.4)$$

Since $g(1/2, 1/2) = 0$ and $g(1, 0) = 1 - q > 0$, it follows that $g(x, 1 - x) > 0$ for all $x \in (1/2, 1]$. Finally, since g is strictly concave on B , the first equality in (3.4) implies that $g(x, y) > \inf_{x \in \{p^*, 1\}} g(x, 1 - x)$ for $(x, y) \in B$, which is (3.3). \square

Equipped with this technical lemma, we can now prove Lemma 3.3.

Proof of Lemma 3.3. We prove the lemma in cases. There are three cases to consider: $p_{j-1}^n < p_j^n$, $p_{j-1}^n = p_j^n$, and $p_{j-1}^n > p_j^n$. In each case, we will use the definition of p_j^{n+1} from (1.14):

$$p_j^{n+1} = p_j^n - q[(p_j^n)^{m+1} - (p_{j-1}^n)^{m+1}]. \quad (3.5)$$

When $p_{j-1}^n < p_j^n$, (3.2) reduces to showing that $p_j^{n+1} < p_j^n$. This is clear from (3.5), since $(p_j^n)^{m+1} - (p_{j-1}^n)^{m+1} > 0$.

Similarly, when $p_{j-1}^n = p_j^n$, (3.2) reduces to showing that $p_j^{n+1} = p_j^n$. Again, from (3.5), we see that $(p_j^n)^{m+1} - (p_{j-1}^n)^{m+1} = 0$, while implies the result.

For the final case, when $p_{j-1}^n > p_j^n$, establishing (3.2) reduces to showing that $p_j^{n+1} < p_{j-1}^n$. By (3.5), this is equivalent to showing that

$$0 < p_{j-1}^n - q(p_{j-1}^n)^{m+1} - (p_j^n - q(p_j^n)^{m+1}).$$

Using again the definition of $g(x, y) = f(x) - f(y)$ as in Lemma 3.4, we can see this is equivalent to showing that

$$g(p_{j-1}^n, p_j^n) > 0.$$

Because $p_{j-1}^n + p_j^n \leq 1$ and $p_{j-1}^n > p_j^n \geq 0$, Lemma 3.4 exactly yields the result. \square

Remark 3.5. Combining Lemma 3.2 and Lemma 3.3, we are able to identify the precise value of Λ which guarantees stochastic monotonicity, as discussed just above the proof of Proposition 2.1. In particular, if F_k^0 and \tilde{F}_k^0 are two CDFs such that $F_k^0 \leq \tilde{F}_k^0$ and

$$0 \leq F_k^0 - F_{k-1}^0 \leq p^* \quad \text{and} \quad 0 \leq \tilde{F}_k^0 - \tilde{F}_{k-1}^0 \leq p^*, \quad (3.6)$$

then Lemma 3.2 guarantees that $F_k^1 \leq \tilde{F}_k^1$, and Lemma 3.3 guarantees that F_k^1 and \tilde{F}_k^1 both satisfy (3.6). We may then conclude (by induction) that $F_k^n \leq \tilde{F}_k^n$ for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, so the CM-evolution is stochastically monotone for the corresponding initial distributions μ and $\tilde{\mu}$.

Now we are ready to prove Proposition 3.1. We will do so by relating the process starting from a p^* -bounded initial condition to a sequence of processes which begin from a discretization of a Lipschitz function. We will then be able to use Proposition 2.1 for Lipschitz continuous initial data to yield convergence in this extended setting.

Proof of Proposition 3.1. Fix $\varepsilon > 0$. Let $(X_n, n \geq 0)$ be $\text{CM}(m, q, \mu)$ -distributed. Then the collection of values $F_k^n = \mathbf{P}(X_n < k)$ satisfy the recursive relationship (2.1) with initial condition $F_k^0 = \mathbf{P}(X_0 < k) = \mu(-\infty, k)$. We note that the values F_k^n also satisfy (2.15) when Δ_x and Δ_t are chosen so that $(\Delta_x)^{m+1} = \Delta_t$. We enforce this relation between Δ_x and Δ_t throughout the proof.

We will sandwich F_k^n between two solutions of (2.15) with smoother initial conditions. To this end, define $u^{\varepsilon,1}$ as the lsc viscosity solution of

$$\begin{cases} u_t^{\varepsilon,1} + q|u_x^{\varepsilon,1}|^{m+1} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u^{\varepsilon,1}(x, 0) = \varepsilon \mathbb{1}_{\{x \leq 0\}} + \mathbb{1}_{\{x > 0\}} & \text{in } \mathbb{R}, \end{cases}$$

and let $u^{0,1-\varepsilon}$ denote the lsc viscosity solution of

$$\begin{cases} u_t^{0,1-\varepsilon} + q|u_x^{0,1-\varepsilon}|^{m+1} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u^{0,1-\varepsilon}(x, 0) = (1 - \varepsilon) \mathbb{1}_{\{x > 0\}} & \text{in } \mathbb{R}. \end{cases}$$

Setting

$$S := S(\varepsilon) = (1 - \varepsilon)^{m/(m+1)} (m+1) q^{1/(m+1)} m^{-m/(m+1)} \varepsilon^{1/(m+1)}, \quad (3.7)$$

by (1.12), these solutions have the explicit forms

$$u^{\varepsilon,1}(x,t) = \begin{cases} \varepsilon & \text{if } x \leq 0, \\ \varepsilon + \frac{m}{q^{\frac{1}{m}}(m+1)^{\frac{m+1}{m}}} \left(\frac{x^{m+1}}{t}\right)^{\frac{1}{m}} & \text{if } 0 \leq x \leq S(\varepsilon)(t/\varepsilon)^{1/(m+1)}, \\ 1 & \text{otherwise.} \end{cases} \quad (3.8)$$

and

$$u^{0,1-\varepsilon}(x,t) = \begin{cases} 0 & \text{if } x \leq 0, \\ \frac{m}{q^{\frac{1}{m}}(m+1)^{\frac{m+1}{m}}} \left(\frac{x^{m+1}}{t}\right)^{\frac{1}{m}} & \text{if } 0 \leq x \leq S(\varepsilon)(t/\varepsilon)^{1/(m+1)}, \\ 1 - \varepsilon & \text{otherwise.} \end{cases} \quad (3.9)$$

In particular, $u^{\varepsilon,1}(x,t) = u^{0,1-\varepsilon}(x,t) + \varepsilon$. We also see from (3.8) and (3.9) that both $u^{\varepsilon,1}(x,\varepsilon)$, $u^{0,1-\varepsilon}(x,\varepsilon)$ are Lipschitz continuous with the same Lipschitz constant $K = K(\varepsilon)$, and therefore, there exists an $\eta = \eta(\varepsilon)$ sufficiently small such that if $\Delta_x \leq \eta$, then

$$\begin{cases} 0 \leq u^{\varepsilon,1}(x + \Delta_x, \varepsilon) - u^{\varepsilon,1}(x, \varepsilon) \leq p^* & \text{for all } x \in \mathbb{R}, \\ 0 \leq u^{0,1-\varepsilon}(x + \Delta_x, \varepsilon) - u^{0,1-\varepsilon}(x, \varepsilon) \leq p^* & \text{for all } x \in \mathbb{R}. \end{cases} \quad (3.10)$$

Also, by our explicit representation of $u^{\varepsilon,1}$ in (3.8), we have

$$\begin{aligned} u^{\varepsilon,1}(x, \varepsilon) &\geq \varepsilon \quad \text{for all } x, \\ u^{\varepsilon,1}(x, \varepsilon) &= 1 \quad \text{for all } x \geq S(\varepsilon). \end{aligned} \quad (3.11)$$

Now, define

$$\begin{aligned} L &:= L(\varepsilon) = \max\{k \leq 0 : F_k^0 \leq \varepsilon\}, \\ R &:= R(\varepsilon) = \min\{k \geq 0 : F_k^0 \geq 1 - \varepsilon\}. \end{aligned}$$

These values are both finite because μ is a probability distribution on \mathbb{Z} and hence $\lim_{k \rightarrow -\infty} F_k^0 = 0$ and $\lim_{k \rightarrow \infty} F_k^0 = 1$. Then, for $\tilde{n} \in \mathbb{N}$, let $F_k^{+,n}(\tilde{n})$ and $F_k^{-,n}(\tilde{n})$ be the schemes defined by (2.15), with $(\Delta_x)^{m+1} = \Delta_t$ as always, and with initial conditions

$$F_k^{+,0}(\tilde{n}) = u^{\varepsilon,1}(k\Delta_x - L\tilde{n}^{-1/(m+1)} + S, \varepsilon) \quad (3.12)$$

and

$$F_k^{-,0}(\tilde{n}) = u^{0,1-\varepsilon}(k\Delta_x - R\tilde{n}^{-1/(m+1)}, \varepsilon), \quad (3.13)$$

respectively. We use the parameter \tilde{n} to spatially shift the initial conditions in order to obtain ordered initial conditions.

By the definition of L , for $k < L$ we have

$$F_k^0 \leq \varepsilon \leq F_k^{+,0}(\tilde{n}),$$

the second inequality since $F_k^{+,0}(\tilde{n}) \geq \varepsilon$ for all k by (3.11). For $k \geq L$, if $\Delta_x \leq \tilde{n}^{-1/(m+1)}$ then since $L \leq 0$ we have $k\Delta_x - L\tilde{n}^{-1/(m+1)} + S \geq S$, so also by (3.11),

$$F_k^{+,0}(\tilde{n}) = u^{\varepsilon,1}(k\Delta_x - L\tilde{n}^{-1/(m+1)} + S, \varepsilon) = 1 \geq F_k^0.$$

Therefore, if $\Delta_x \leq \tilde{n}^{-1/(m+1)}$ then $F_k^{+,0}(\tilde{n}) \geq F_k^0$ for all $k \in \mathbb{Z}$.

Similarly, by the definition of R and (3.9), for $k > R$, we have

$$F_k^{-,0}(\tilde{n}) \leq 1 - \varepsilon \leq F_k^0,$$

and if $\Delta_x \leq \tilde{n}^{-1/(m+1)}$, then for $k \leq R$ we have $k\Delta_x - R\tilde{n}^{-1/(m+1)} \leq 0$, so by (3.9),

$$F_k^{-,0}(\tilde{n}) = 0 \leq F_k^0.$$

Thus if $\Delta_x \leq \tilde{n}^{-1/(m+1)}$ then $F_k^{-,0}(\tilde{n}) \leq F_k^0$ for all $k \in \mathbb{Z}$.

Combining the two preceding paragraphs, we obtain that if $\Delta_x \leq \tilde{n}^{-1/(m+1)}$ then for all k ,

$$F_k^{-,0}(\tilde{n}) \leq F_k^0 \leq F_k^{+,0}(\tilde{n}).$$

If also $\Delta_x \leq \eta$, then by (3.10), each scheme satisfies the condition that

$$|F_k^{\pm,0}(\tilde{n}) - F_{k-1}^{\pm,0}(\tilde{n})| \leq p^*$$

for all $k \in \mathbb{Z}$. By Remark 3.5, the prior two displays yield that whenever $\Delta_x \leq \min(\eta, 1/\tilde{n}^{(m+1)})$, we have by induction that for all $n \in \mathbb{N}$,

$$F_k^{-,n}(\tilde{n}) \leq F_k^n \leq F_k^{+,n}(\tilde{n}). \quad (3.14)$$

We now combine these bounds with Proposition 2.1. We first aim to apply the proposition with μ_N defined by

$$\mu_N[-\infty, k) = F_k^{+,0}(\tilde{n}).$$

The proposition requires that μ_N have the form $\mu_N[-\infty, k) = u_0(k/N^{1/(m+1)})$, so the definition of $F_k^{+,0}(\tilde{n})$ forces us to take $\Delta_x = N^{-1/(m+1)}$ and $u_0(x) = u^{\varepsilon,1}(x - L/\tilde{n}^{1/(m+1)} + S, \varepsilon)$. Since $u^{\varepsilon,1}$ is Lipschitz, fixing $T > 1$ and applying Proposition 2.1 (specifically (2.3)) at time $t = 1 \in [0, T]$, it follows that there exist $N_0 = N_0(q, m, K)$ and $c = c(K, m, T)$ such that if $N \geq N_0$, for all $k \in \mathbb{Z}$,

$$F_k^{+,N}(\tilde{n}) \leq u^{\varepsilon,1}(kN^{-1/(m+1)} - L\tilde{n}^{-1/(m+1)} + S, 1 + \varepsilon) + cN^{-1/2}.$$

We emphasize that N_0 and c depend only on the initial condition $u_0(x) = u^{\varepsilon,1}(x - L/\tilde{n}^{1/(m+1)} + S, \varepsilon)$ through its Lipschitz constant K ; in particular, N_0 and c do not depend on \tilde{n} since varying \tilde{n} translates the initial condition horizontally but does not change its Lipschitz constant.

Similarly, taking $\Delta_x = N^{-1/(m+1)}$ and $u_0(x) = u^{0,1-\varepsilon}(x - R/\tilde{n}^{1/(m+1)}, \varepsilon)$ and $t = 1$, applying Proposition 2.1 (specifically (2.3)) with μ_N defined by $\mu_N[-\infty, k) = F_k^{-,0}(\tilde{n})$ yields that for all $N \geq N_0$ and all $k \in \mathbb{Z}$,

$$F_k^{-,N}(\tilde{n}) \geq u^{0,1-\varepsilon}(kN^{-1/(m+1)} - R\tilde{n}^{-1/(m+1)}, 1 + \varepsilon) - cN^{-1/2}.$$

For $N \geq N_0$ large enough that also $\Delta_x = N^{-1/(m+1)} \leq \min(\eta, \tilde{n}^{-1/(m+1)})$, we may combine these bounds with (3.14) to deduce that for all $k \in \mathbb{Z}$,

$$\mathbf{P}(X_N < k) = F_k^N \leq F_k^{+,N}(\tilde{n}) \leq u^{\varepsilon,1}(kN^{-1/(m+1)} - L\tilde{n}^{-1/(m+1)} + S, 1 + \varepsilon) + cN^{-1/2}$$

and

$$\mathbf{P}(X_N < k) = F_k^N \geq F_k^{-,N}(\tilde{n}) \geq u^{0,1-\varepsilon}(kN^{-1/(m+1)} - R\tilde{n}^{-1/(m+1)}, 1 + \varepsilon) - cN^{-1/2}.$$

Taking $k = xN^{1/(m+1)}$, these bounds become

$$\mathbf{P}(X_N < xN^{1/(m+1)}) \leq u^{\varepsilon,1}(x - L\tilde{n}^{-1/(m+1)} + S, 1 + \varepsilon) + cN^{-1/2},$$

and

$$\mathbf{P}(X_N < xN^{1/(m+1)}) \geq u^{0,1-\varepsilon}(x - R\tilde{n}^{-1/(m+1)}, 1 + \varepsilon) - cN^{-1/2}.$$

We should in fact take $k = \lfloor xN^{1/(m+1)} \rfloor$ above, but we ignore this minor rounding issue to preserve readability, as the errors it creates are asymptotically negligible for N large due to the spatial continuity of $u^{\varepsilon,1}$ and of $u^{0,1-\varepsilon}$ at time $1 + \varepsilon$.

For $\tilde{n} \geq \max(N_0, \eta^{m+1})$, if $N \geq \tilde{n}$ then the other constraints on N are automatically satisfied. Recalling that $L \leq 0$, since $cN^{-1/2} \rightarrow 0$ as $N \rightarrow \infty$, the first of the two

preceding bounds then implies that

$$\begin{aligned} \limsup_{N \rightarrow \infty} \mathbf{P} \left(\frac{X_N}{N^{1/(m+1)}} < x \right) &\leq \inf_{\tilde{n} \geq \max(N_0, \eta^{m+1})} \left(u^{\varepsilon,1}(x - L\tilde{n}^{-1/(m+1)} + S, 1 + \varepsilon) \right) \\ &= u^{\varepsilon,1}(x + S, 1 + \varepsilon). \end{aligned}$$

Likewise, the second of the bounds yields that

$$\begin{aligned} \liminf_{N \rightarrow \infty} \mathbf{P} \left(\frac{X_N}{N^{1/(m+1)}} < x \right) &\geq \sup_{\tilde{n} \geq \max(N_0, \eta^{m+1})} \left(u^{0,1-\varepsilon}(x - R\tilde{n}^{-1/(m+1)}, 1 + \varepsilon) \right) \\ &= u^{0,1-\varepsilon}(x, 1 + \varepsilon). \end{aligned}$$

Finally, from the explicit representations of $u^{\varepsilon,1}, u^{0,1-\varepsilon}$ from (3.8) and (3.9), and $S = S(\varepsilon)$ defined by (3.7), we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} S(\varepsilon) &= 0 \\ \lim_{\varepsilon \rightarrow 0} u^{\varepsilon,1}(x + S(\varepsilon), 1 + \varepsilon) &= u(x, 1) \\ \lim_{\varepsilon \rightarrow 0} u^{0,1-\varepsilon}(x, 1 + \varepsilon) &= u(x, 1), \end{aligned}$$

uniformly in x , where $u(x, t)$ is given by (1.13). Taking the limit as $\varepsilon \rightarrow 0$, we get that

$$u(x, 1) \leq \liminf_{n \rightarrow \infty} \mathbf{P} \left(\frac{X_n}{n^{1/(m+1)}} < x \right) \leq \limsup_{n \rightarrow \infty} \mathbf{P} \left(\frac{X_n}{n^{1/(m+1)}} < x \right) \leq u(x, 1),$$

and therefore

$$\lim_{n \rightarrow \infty} \mathbf{P} \left(\frac{X_n}{n^{1/(m+1)}} < x \right) = u(x, 1),$$

as desired. \square

4. GENERAL SINGULAR INITIAL CONDITIONS

We saw that Lemma 3.2 requires a bound on the maximum single-site probability. Our next result shows that in fact, there exists a constant N_1 such that, regardless of the initial distribution, the distribution of the CM after N_1 steps will satisfy such a bound.

Lemma 4.1. *Let μ be a probability distribution with support \mathbb{Z} , and let $(X_n, n \geq 0)$ be $\text{CM}(m, q, \mu)$ -distributed. Then there exists a constant $N_1 = N_1(q, m)$, such that for all $n \geq N_1$,*

$$\max_{k \in \mathbb{Z}} \mathbf{P}(X_n = k) \leq p^*. \quad (4.1)$$

Proof of Lemma 4.1. First, by Lemma 3.3, if $M := \max_{k \in \mathbb{Z}} p_k^0 = \max_{k \in \mathbb{Z}} \mu(\{k\}) \leq p^*$ then $\max_{k \in \mathbb{Z}} p_k^1 \leq p^*$. Therefore, it suffices to show that there exists $C = C(m, q) > 0$ such that if $M > p^*$ then $\max_{k \in \mathbb{Z}} p_k^1 \leq M - C$, since then by induction, for all $n \in \mathbb{N}$,

$$\max_{k \in \mathbb{Z}} p_k^n \leq \max(p^*, M - nC),$$

and in particular $\max_{k \in \mathbb{Z}} p_k^n \leq p^*$ for all $n \geq (1 - p^*)/C$.

So suppose $M > p^*$. Since $p^* > 1/2$, there is a unique integer $\ell \in \mathbb{Z}$ with $p_\ell^0 > p^*$. We bound p_k^1 for all $k \in \mathbb{Z}$ by splitting into three cases, according to whether $k = \ell$, $k = \ell + 1$ or $k \notin \{\ell, \ell + 1\}$.

- If $k \notin \{\ell, \ell + 1\}$, then by Lemma 3.3 we have

$$p_k^1 \leq \max(p_k^0, p_{k-1}^0) \leq \max_{k \in \mathbb{Z}, k \neq \ell} p_k^0 \leq 1/2 < M - (p^* - 1/2).$$

- If $k = \ell$, then by (3.5) and since $p_k^0 = M > p^*$ and $p_{k-1}^0 < 1/2$, we have

$$\begin{aligned} p_k^1 &= p_k^0 - q[(p_k^0)^{m+1} - (p_{k-1}^0)^{m+1}] \leq p_k^0 - q[(p^*)^{m+1} - (1/2)^{m+1}] \\ &= M - q[(p^*)^{m+1} - (1/2)^{m+1}]. \end{aligned}$$

- If $k = \ell + 1$, then we use Lemma 3.4, which provides lower bounds on the function $g(x, y) = (x - qx^{m+1}) - (y - qy^{m+1})$. Note that

$$g(p_{k-1}^0, p_k^0) = p_{k-1}^0 - q(p_{k-1}^0)^{m+1} - p_k^0 + q(p_k^0)^{m+1} = p_{k-1}^0 - p_k^1.$$

Since $p_{k-1}^0 = p_\ell^0 = M > p^*$ and $p_{k-1}^0 + p_k^0 \leq 1$, Lemma 3.4 and the preceding displayed identity together imply that

$$p_k^1 = p_{k-1}^0 - g(p_{k-1}^0, p_k^0) \leq M - \min(g(p^*, 1 - p^*), 1 - q).$$

Taking $C_1 = p^* - 1/2$, $C_2 = q[(p^*)^{m+1} - (1/2)^{m+1}]$, $C_3 = \min(g(p^*, 1 - p^*), 1 - q)$, and $C = \min(C_1, C_2, C_3)$, the above bounds then give that $\max_{k \in \mathbb{Z}} p_k^1 \leq M - C$, as required. \square

We can combine this result with Proposition 3.1 to complete the proof of the main theorem.

Proof of Theorem 1.1. Fix $N_1 = N_1(q, m)$ as in Lemma 4.1, let $\tilde{\mu}$ be the distribution of X_{N_1} and let $\tilde{X}_n = X_{N_1+n}$ for $n \geq 0$. Then $(\tilde{X}_n, n \geq 0)$ is $\text{CM}(m, q, \tilde{\mu})$ -distributed. Because $\mathbf{P}(\tilde{X}_0 = k) \leq p^*$ for all k , we can apply Proposition 3.1 to conclude that

$$\lim_{n \rightarrow \infty} \mathbf{P}\left(\frac{\tilde{X}_n}{n^{1/(m+1)}} < x\right) = u(x, 1).$$

Since N_1 is fixed and $u(x, 1)$ is continuous, this implies that for all $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbf{P}\left(\frac{X_n}{n^{1/(m+1)}} < x\right) = \lim_{n \rightarrow \infty} \mathbf{P}\left(\frac{\tilde{X}_n}{n^{1/(m+1)}} < x \left(\frac{n - N_1}{n}\right)^{1/(m+1)}\right) = u(x, 1). \quad (4.2)$$

By comparing the expression for $u(x, 1)$ provided by (1.13) to the CDF given in (1.3) for $(m+1) \left(\frac{q}{m^m}\right)^{1/m+1} B$, where B is $\text{Beta}(\frac{m+1}{m}, 1)$ -distributed, we see that

$$\frac{1}{m+1} \left(\frac{m^m}{q}\right)^{\frac{1}{m+1}} \cdot \frac{X_n}{n^{1/(m+1)}} \xrightarrow{d} B,$$

as required. \square

5. GENERALIZATIONS, LIMITATIONS, AND OPEN QUESTIONS

In this section, we discuss several possible extensions to the above results, as well as some obstacles and challenges we have observed.

1. Higher Dimensions. One may try to extend our results and techniques to higher space-dimensions. However, there are several challenges. In particular, we use the monotonicity of CDFs, namely that F_k^n is nondecreasing in k , ubiquitously throughout the paper. The monotonicity of CDFs in higher dimensions is weaker, as it requires ordering in all coordinates. Relatedly, it is unclear how to extend the stochastic monotonicity and sandwiching arguments from this paper to higher dimensions. These points make generalization of our approach to dimensions $d > 1$ delicate (although we hope not impossible).

2. Cooperative Motion with fewer than m friends. A related model which we have not considered, but which may be amenable to the techniques of this paper, is when the cooperative motion only requires ℓ individuals to move, for $\ell < m$. More precisely, we

may modify the model as follows. Let X_0 and $(D_n, n \geq 0)$ be as in the introduction. Then, for $n \geq 0$, let $(\tilde{X}_n^i, 1 \leq i \leq m)$ be independent copies of X_n , and set

$$X_{n+1} = \begin{cases} X_n + D_n & \text{if } X_n = \tilde{X}_n^i \text{ for at least } \ell \text{ distinct values } i \in \{1 \dots m\}, \\ X_n & \text{otherwise.} \end{cases}$$

It seems likely that for such a process, X_n should typically take values of order $n^{1/(\ell+1)}$. A heuristic argument for this is as follows. Suppose that X_n/n^α behaves roughly like a continuous random variable with compact support, for n large; say that $\mathbf{P}(X_n = k) \asymp n^{-1/\alpha}$ for $\Theta(n^\alpha)$ distinct values of k , and for other values of k this probability is substantially smaller.

On one hand, this suggests that $\mathbf{P}(X_{n+1} > X_n) = \Theta(n^{\alpha-1})$, since we expect that $X_{2n} - X_n = \Theta(n^\alpha)$. On the other hand, $\mathbf{P}(X_{n+1} > X_n)$ is the probability that at least ℓ of the m copies of \tilde{X}_n^i take the same value as X_n ; if the distribution of X_n is spread out over roughly n^α sites, then this probability should be around $(n^{-\alpha})^\ell$. For these two predictions to agree we must have $1 - \alpha = \alpha\ell$, so $\alpha = 1/(\ell + 1)$.

While we have confidence in this prediction of the asymptotic size of X_n , it is not clear to us whether or not the scaling limit should in fact be the same as for a $\text{CM}(\ell, q, \mu)$ process.

3. Cooperation Motion with a non-integer number of friends. Another possible extension is to the case when m is non-integer. Although m integer has a natural interpretation in terms of cooperative motion, which in turn leads to the recursion relation (1.4), we may alternatively take (1.4) as a definition for the CDF F_k^n of a random variable X_n . In this case, the same analysis shows that for any $m \in \mathbb{R}$ with $m \geq 1$, Theorem 1.1 still holds when X_0 is μ -distributed and $\mathbf{P}(X_n < k) = F_k^n$, where F_k^n is defined according to (1.4).

The requirement that $m \geq 1$ is crucial for Lemma 3.4, so the techniques of this paper do not yield insight into what happens for $m \in (0, 1)$. However, it would be quite interesting to understand this, as well as the limiting behaviour as $m \rightarrow 0$.

General step size distributions. Remaining in one spatial dimension, another natural generalization of this process would be to consider cooperative motion which allowed for more general step sizes D_n . As we will discuss in Section 5.2, it turns out that if $\mathbf{P}(|D_n| > 1) > 0$, we confront an immediate, provable obstacle to directly applying the proof techniques of this paper (failure of monotonicity). However, before describing this obstacle, we first present a generalization of our main result. If the steps $(D_n, n \geq 0)$ are an integer multiple of Bernoulli(q) random variables, then we are able to prove a distributional convergence result; this is presented in the next subsection. Surprisingly, the limiting distribution in this case, although always a mixture of Beta random variables, need not be Beta-distributed, due to lattice effects which persist at large times.

5.1. Persistent lattice effects. Let $(D_n, n \geq 0)$ be iid, non-negative, bounded integer random variables. Define a cooperative motion process, with X_0 chosen according to an initial probability distribution μ on \mathbb{Z} , as follows. For $n \geq 0$, let $(\tilde{X}_n^i, 1 \leq i \leq m)$ be independent copies of X_n , and set

$$X_{n+1} = \begin{cases} X_n + D_n & \text{if } X_n = \tilde{X}_n^i \text{ for all } i = 1 \dots m, \\ X_n & \text{if } X_n \neq \tilde{X}_n^i \text{ for some } i. \end{cases} \quad (5.1)$$

In this section we consider the case where the steps $(D_n, n \geq 0)$ are iid with $\mathbf{P}(D_n = g) = q = 1 - \mathbf{P}(D_n = 0)$ for some $q \in (0, 1)$ and $g \in \mathbb{N}$. If the initial distribution μ is supported by a translate of $g\mathbb{Z}$ then the resulting cooperative motion may simply

be seen as a rescaling of the Bernoulli cooperative motion process considered in the body of the paper. However, if μ is not supported by a translate of $g\mathbb{Z}$ then the asymptotic behaviour is in fact different; there are lattice effects which persist at large times.

Theorem 5.1. *Consider the generalized cooperative motion with $\mathbf{P}(D_n = g) = q = 1 - \mathbf{P}(D_n = 0)$ for some $q \in (0, 1)$ and $g \in \mathbb{N}$. Write $\pi_r = \mathbf{P}(X_0 = r \bmod g)$ for $r \in \{1, 2, \dots, g\}$. Then*

$$\frac{1}{g} \frac{1}{m+1} \left(\frac{m^m}{qn} \right)^{1/(m+1)} X_n \xrightarrow{d} B \cdot \sum_{r=1}^g (\pi_r)^{m/(m+1)} \mathbb{1}_{\{A=r\}},$$

where A is a random variable taking values in $\{1, 2, \dots, g\}$ with $\mathbf{P}(A = r) = \pi_r$, and B is Beta($\frac{m+1}{m}, 1$)-distributed and independent of A .

As an input to the proof of Theorem 5.1, we use the following straightforward extension of Theorem 1.1 to Bernoulli cooperative motions which may take values $\pm\infty$. Let $c = c(q, m) = mq^{-1/m}(m+1)^{-(m+1)/m}$, and for $0 \leq a < b \leq 1$ define an extended CDF $F^{a,b}$ by

$$F^{a,b}(x) = \begin{cases} a & \text{if } x \leq 0 \\ a + cx^{(m+1)/m} & \text{if } 0 \leq cx^{(m+1)/m} \leq b-a \\ b & \text{if } b-a \leq cx^{(m+1)/m}. \end{cases}$$

Let $B^{a,b}$ be an extended random variable with distribution $F^{a,b}$. Then $\mathbf{P}(|B^{a,b}| < \infty) = b-a$, and for $x \in \mathbb{R}$,

$$\mathbf{P}(B^{a,b} \leq x \mid |B^{a,b}| < \infty) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{c}{b-a} x^{(m+1)/m} & \text{if } 0 \leq \frac{c}{b-a} x^{(m+1)/m} \leq 1 \\ 1 & \text{if } 1 \leq \frac{c}{b-a} x^{(m+1)/m}. \end{cases}$$

In other words, given that $|B^{a,b}|$ is finite, it is distributed as $(\frac{b-a}{c})^{m/(m+1)} B$ where B is Beta($\frac{m+1}{m}, 1$)-distributed

Proposition 5.2. *If μ is a probability distribution on $\mathbb{Z} \cup \{\pm\infty\}$ with $\mu(\{-\infty\}) = a$ and $\mu(\{+\infty\}) = 1-b$, and $(X_n, n \geq 0)$ is CM(m, q, μ)-distributed, then*

$$\frac{X_n}{n^{1/(m+1)}} \xrightarrow{d} B^{a,b}.$$

The proof of Proposition 5.2 proceeds exactly as does the proof of Theorem 1.1, with minor notational changes, so we omit the details.

Corollary 5.3. *Suppose that μ is a probability distribution on $\mathbb{Z} \cup \{\pm\infty\}$ with $\mu(\{-\infty\}) = a$ and $\mu(\{+\infty\}) = 1-b$, and that $\mathbf{P}(D_n = g) = q = 1 - \mathbf{P}(D_n = 0)$ for some $q \in (0, 1)$ and $g \in \mathbb{N}$, $g > 0$. If there is $r \in \{1, 2, \dots, g\}$ such that $\mathbf{P}(|X_0| = r \bmod g \mid |X_0| < \infty) = 1$, then*

$$\frac{1}{g} \cdot \frac{X_n}{n^{1/(m+1)}} \xrightarrow{d} B^{a,b}.$$

Proof. Apply Proposition 5.2 to the process $((X_n - r)/g, n \geq 0)$. □

Proof of Theorem 5.1. Define auxiliary processes $(X_n^{(r)}, n \geq 0)$ for $1 \leq r \leq g$ by

$$X_n^{(r)} = \begin{cases} X_n & \text{if } X_n = r \bmod g \\ -\infty & \text{otherwise.} \end{cases}$$

Then by Corollary 5.3, for each $1 \leq r \leq g$,

$$\frac{1}{g} \cdot \frac{X_n^{(r)}}{n^{1/(m+1)}} \xrightarrow{d} B^{1-\pi_r, 1}.$$

Moreover, since exactly one of $X_n^{(1)}, \dots, X_n^{(g)}$ is finite, and $\mathbf{P}(|X_n^{(r)}| < \infty) = \pi^{(r)}$ for all $n \geq 0$ and $1 \leq r \leq g$, it follows that

$$\left(\frac{1}{g} \cdot \frac{X_n^{(r)}}{n^{1/(m+1)}}, 1 \leq r \leq g \right) \xrightarrow{d} (B^{1-\pi_r, 1}, 1 \leq r \leq g),$$

where the joint distribution of the variables on the right-hand side is fully determined by the stipulation that exactly one of them is finite and all others take the value $-\infty$.

Finally, with the convention that $(-\infty) \cdot 0 = 0$, we have

$$X_n = \sum_{r=1}^g X_n^{(r)} \mathbf{1}_{\{|X_n^{(r)}| < \infty\}},$$

together with which the preceding joint convergence implies that

$$\frac{1}{g} \frac{X_n}{n^{1/(m+1)}} \xrightarrow{d} \sum_{r=1}^g B^{1-\pi_r, 1} \mathbf{1}_{\{|B^{1-\pi_r, 1}| < \infty\}}.$$

Since $\mathbf{P}(|B^{1-\pi_r, 1}| < \infty) = 1 - (1 - \pi_r) = \pi_r$ for each $r \in \{1, 2, \dots, g\}$, and the conditional distribution of $B^{1-\pi_r, 1}$ given that $|B^{1-\pi_r, 1}|$ is finite is that of $(\pi_r/c)^{m/(m+1)} \text{Beta}(\frac{m+1}{m}, 1)$, the result follows. \square

5.2. Step Sizes $|D_n| > 1$. Building on our main theorem, and in view of the persistent lattice effects explained in the preceding subsection, we make the following conjecture. Consider the generalized cooperative motion defined by (5.1) and write ν for the common distribution of $(D_n, n \geq 0)$. If $\gcd(k > 0 : \mathbf{P}(D_n = k) > 0) = 1$, then there exists $c = c(\nu) > 0$ such that $cn^{-1/(m+1)}X_n \xrightarrow{d} B$, where B is $\text{Beta}(\frac{m+1}{m}, 1)$ -distributed.

The preceding conjecture states that all totally asymmetric cooperative motion processes with non-negative, bounded integer step sizes whose support is not contained in a proper sublattice of \mathbb{Z} should have similar asymptotic behaviour. However, there is a provable difficulty in establishing this conjecture beyond the Bernoulli setting using the proof techniques shown above. Specifically, we next show that monotonicity of the evolution fails to hold whenever $\mathbf{P}(|D_n| > 1) > 0$. This implies that, in some sense, the main proof technique used in this paper can only handle cooperative motion-type processes with $|D_n| \leq 1$.

Consider a cooperative-motion type process as in (5.1), with bounded but not necessarily positive step sizes, so $\mathbf{P}(-\ell \leq D_n \leq s) = 1$ for some non-negative integers s and ℓ . Writing $F_k^n = \mathbf{P}(X_n < k)$, then the values F_k^n satisfy the following recurrence:

$$\begin{aligned} F_k^{n+1} &= G(F_{k+\ell}^n, \dots, F_k^n, \dots, F_{k-s}^n) \\ &:= F_k^n - \sum_{j=k-s}^{k-1} \mathbf{P}(X_n = j)^{m+1} \mathbf{P}(D_n \geq k-j) + \sum_{j=k}^{k+\ell-1} \mathbf{P}(X_n = j)^{m+1} \mathbf{P}(D_n < k-j) \\ &= F_k^n - \sum_{j=k-s}^{k-1} (F_{j+1}^n - F_j^n)^{m+1} \mathbf{P}(D_n \geq k-j) + \sum_{j=k}^{k+\ell-1} (F_{j+1}^n - F_j^n)^{m+1} \mathbf{P}(D_n < k-j). \end{aligned}$$

The function G is defined by the equality of the first and third lines, above: so

$$\begin{aligned} G(f_{k+\ell}, \dots, f_{k-s}) \\ = f_k - \sum_{j=k-s}^{k-1} (f_{j+1} - f_j)^{m+1} \mathbf{P}(D_n \geq k-j) + \sum_{j=k}^{k+\ell-1} (f_{j+1} - f_j)^{m+1} \mathbf{P}(D_n < k-j). \end{aligned}$$

Theorem 5.4. *If $\mathbf{P}(|D_n| > 1) > 0$, then there is no $\Lambda > 0$ such that G is non-decreasing in each argument whenever*

$$0 \leq f_{j+1} - f_j \leq \Lambda$$

for all $j \in [k-s, k+\ell-1]$.

Proof. First,

$$\begin{aligned} \frac{\partial G}{\partial f_{k-1}} &= (m+1)(f_k - f_{k-1})^m \mathbf{P}(D_n \geq k - (k-1)) \\ &\quad - (m+1)(f_{k-1} - f_{k-2})^m \mathbf{P}(D_n \geq k - (k-2)) \\ &= (m+1)(f_k - f_{k-1})^m \mathbf{P}(D_n \geq 1) - (m+1)(f_{k-1} - f_{k-2})^m \mathbf{P}(D_n \geq 2), \end{aligned}$$

so if $\mathbf{P}(D_n \geq 2) > 0$ and if $f_k = f_{k-1} > f_{k-2}$, then

$$\frac{\partial G}{\partial f_{k-1}} < 0.$$

Similarly,

$$\frac{\partial G}{\partial f_{k+1}} = (m+1)(f_{k+1} - f_k)^m \mathbf{P}(D_n < 0) - (m+1)(f_{k+2} - f_{k+1})^m \mathbf{P}(D_n < -1),$$

so if $\mathbf{P}(D_n < -1) > 0$, then whenever $f_k = f_{k+1} < f_{k+2}$ then we have

$$\frac{\partial G}{\partial f_{k+1}} < 0. \quad \square$$

Note that for any initial distribution with bounded support, if the step size is bounded then for all n the support of X_n is bounded: letting $k = \max\{\ell : F_\ell^n < 1\} + 2$ and $k' = \min\{\ell : F_\ell^n > 0\} - 2$, then k and k' are both finite. Moreover, $F_{k-2}^n < F_{k-1}^n = F_k^n = 1$ and $0 = F_{k'}^n = F_{k'+1}^n < F_{k'+2}^n$, and thus if $\mathbf{P}(|D_n| > 1) > 0$ then by the above theorem, at no point in the evolution will the process reach a time at which monotonicity can be invoked. Without monotonicity, we can not apply the Crandall-Lions methodology, so the proof technique used in this paper fails.

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APPENDIX A. AN INTRODUCTION TO VISCOSITY SOLUTIONS

In this section, we provide a self-contained description of Crandall-Lions (continuous) and Barron-Jensen (lsc) viscosity solutions. The results of this section are classical and can be found in various references such as [6, 10, 2, 3].

We will work throughout this section with the model equation

$$u_t + H(u_x) = 0, \tag{A.1}$$

where $H : \mathbb{R} \rightarrow \mathbb{R}$. We also define the Cauchy problem, given by

$$\begin{cases} u_t + H(u_x) = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}. \end{cases} \quad (\text{A.2})$$

We begin with the theory of continuous viscosity solutions.

Definition A.1. Let $u : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$. We say that u is a viscosity *subsolution* of (A.1) at (x_0, t_0) if u is upper semicontinuous at (x_0, t_0) , and for any function $\varphi \in C^1(\mathbb{R} \times (0, \infty))$ such that $u - \varphi$ has a local maximum at (x_0, t_0) , we have

$$\varphi_t(x_0, t_0) + H(\varphi_x(x_0, t_0)) \leq 0.$$

We say that u is a viscosity *supersolution* of (A.1) at (x_0, t_0) if u is lower semicontinuous at (x_0, t_0) , and for any function $\varphi \in C^1(\mathbb{R} \times (0, \infty))$ such that $u - \varphi$ has a local minimum at (x_0, t_0) , we have

$$\varphi_t(x_0, t_0) + H(\varphi_x(x_0, t_0)) \geq 0.$$

Finally, we say that u is a viscosity solution of (A.2) if and only if u is both a viscosity subsolution and supersolution of (A.1) for all $(x_0, t_0) \in \mathbb{R} \times (0, \infty)$ and, additionally, for all $x \in \mathbb{R}$, $u(y, t) \rightarrow u_0(x)$ as $(y, t) \rightarrow (x, 0)$. As u is then both upper and lower-semicontinuous on $\mathbb{R} \times (0, \infty)$, u is necessarily continuous.

One can also interpret the definition of viscosity solutions from a geometric perspective. The condition that $u - \varphi$ has a local max/min at (x_0, t_0) can always be replaced by the function φ touching u at the point (x_0, t_0) from above/below. Indeed, when $u - \varphi$ has a local max at (x_0, t_0) , we may adjust φ (adding appropriate constants and strictly convex/concave functions) to obtain $\tilde{\varphi}$ such that

$$u < \tilde{\varphi} \quad \text{in } \mathbb{R} \times (0, \infty), \text{ except at } (x_0, t_0) \text{ where } u(x_0, t_0) = \tilde{\varphi}(x_0, t_0).$$

If u is differentiable at (x_0, t_0) and satisfies (A.1) at (x_0, t_0) , then u automatically satisfies (A.1) in the viscosity sense at (x_0, t_0) . The notion of viscosity solution entails that if u is not differentiable at (x_0, t_0) , one uses a smooth test function which ‘‘touches’’ u at the point (x_0, t_0) on either side to evaluate the PDE at (x_0, t_0) . Compared to other notions of weak solutions of PDEs (for example, distributional solutions which are based on integration by parts), viscosity solutions are particularly amenable to *nonlinear* PDEs. We now recall the basic existence and uniqueness result for continuous viscosity solutions which we will use throughout the paper:

Theorem A.2. [7, Theorem VI.2] *Consider (A.2) with H continuous and u_0 bounded and uniformly continuous. There exists a unique continuous viscosity solution u of (A.2). Moreover,*

$$|u(x, t) - u(y, t)| \leq \sup_{\zeta \in \mathbb{R}} |u_0(\zeta) + u_0(\zeta + y - x)| \quad \text{for } x, y \in \mathbb{R}, t \geq 0.$$

It is well known (see for example [9, 10.3, Theorem 3] that when $H(p)$ is convex and $\lim_{|p| \rightarrow \infty} \frac{H(p)}{|p|} = +\infty$, the unique continuous viscosity solution is given by the Hopf-Lax Formula

$$u(x, t) = \inf_{y \in \mathbb{R}} \left\{ u_0(y) + tH^* \left(\frac{x - y}{t} \right) \right\}.$$

The crown jewel of continuous viscosity solutions theory is the celebrated comparison principle, which is an extremely useful tool for analysis:

Theorem A.3. [6, Theorem 8.2] *Consider (A.2) with H continuous. If u is a subsolution of (A.1) and v is a supersolution of (A.1), and $u(x, 0) = u_0(x) \leq v_0(x) = v(x, 0)$ with u_0, v_0 bounded and uniformly continuous, then $u(x, t) \leq v(x, t)$ for all $t > 0$.*

Using the Comparison Principle (Theorem A.3), we can show that u solving (A.2) satisfies additional regularity estimates:

Proposition A.4. *Let u denote the unique continuous viscosity solution of (A.2) with u_0 bounded and Lipschitz continuous with Lipschitz constant $K > 0$. Then there exists $C > 0$ such that for all $(x, t) \in \mathbb{R} \times (0, \infty)$,*

$$\begin{cases} |u_t(x, t)| \leq C, \\ |u_x(x, t)| \leq K. \end{cases}$$

Proof. The fact that $|u_x| \leq K$ in all of $\mathbb{R} \times (0, \infty)$ is automatic by Theorem A.2. We now show that u_t is uniformly bounded. In order to do so, we note that for $C := \sup_{|p| \leq K} H(p)$,

$$v(x, t) := u_0(x) + Ct \quad \text{and} \quad w(x, t) := u_0(x) - Ct$$

are both super and subsolutions of (A.2) respectively. Therefore, the Comparison Principle (Theorem A.3) yields

$$u_0(x) - Ct \leq u(x, t) \leq u_0(x) + Ct,$$

which implies that

$$\sup_{t>0} \left| \frac{u(x, t) - u_0(x)}{t} \right| \leq C, \tag{A.3}$$

for all $x \in \mathbb{R}$. Now, for any $s > 0$, considering the function $u^s(x, t) := u(x, t + s)$, we have that

$$u^s(x, 0) - \|u(x, 0) - u^s(x, 0)\|_{L^\infty} \leq u(x, 0) \leq u^s(x, 0) + \|u(x, 0) - u^s(x, 0)\|_{L^\infty}.$$

Another application of the Comparison Principle (Theorem A.3) implies that

$$u^s(x, t) - \|u(x, 0) - u^s(x, 0)\|_{L^\infty} \leq u(x, t) \leq u^s(x, t) + \|u(x, 0) - u^s(x, 0)\|_{L^\infty},$$

so that by (A.3),

$$|u(x, t + s) - u(x, t)| \leq \|u(x, 0) - u^s(x, 0)\|_{L^\infty} \leq Cs.$$

This implies that $|u_t| \leq C$ for all $(x, t) \in \mathbb{R} \times (0, \infty)$. \square

We now introduce the notion of Barron-Jensen or lower semicontinuous viscosity solutions, which is only defined when H is a convex function.

Definition A.5. A lower semicontinuous function $u : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ is an lsc viscosity solution of (A.1) at (x_0, t_0) if for every $\varphi \in C^1(\mathbb{R} \times (0, \infty))$ such that $u - \varphi$ has a local minimum at (x_0, t_0) , we have that

$$\varphi_t(x_0, t_0) + H(\varphi_x(x_0, t_0)) = 0.$$

We say that u is a lsc solution of (A.2) if u is a lsc viscosity solution for all $(x_0, t_0) \in \mathbb{R} \times (0, \infty)$ and

$$\inf \left\{ \liminf_{n \rightarrow \infty} u(x_n, t_n) \mid t_n \rightarrow 0, x_n \rightarrow x \right\} = u_0(x).$$

In the case when u is continuous, we have an equivalence between the two definitions:

Theorem A.6. [3, Theorem 16] *Assume H is convex. A continuous function is a viscosity solution of (A.1) if and only if it is a lsc viscosity solution (A.1).*

Finally, we recall that in the case when H is convex, a natural candidate for a solution (from the point of view of optimal control) is the solution given by the Hopf-Lax formula. In the cases when u_0 is lower-semicontinuous and bounded below, the Hopf-Lax formula gives rise to the unique lsc viscosity solution.

Theorem A.7. [2, Theorem 5.2] *Let $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ be lsc with*

$$u_0(x) \geq -C(|x| + 1).$$

Let $H : \mathbb{R} \rightarrow \mathbb{R}$ be convex and Lipschitz. Then

$$u(x, t) = \inf_{y \in \mathbb{R}} \left\{ u_0(y) + tH^* \left(\frac{x-y}{t} \right) \right\}$$

is the unique lsc viscosity solution of (A.2) bounded from below by a function of linear growth.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY
Email address: `louigi.addario@mcgill.ca`

DEPARTMENT OF MATHEMATICS AND STATISTICS, CONCORDIA UNIVERSITY
Email address: `erin.beckman@concordia.ca`

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY
Email address: `jessica.lin@mcgill.ca`