Acyclic improper colourings of graphs with bounded maximum degree

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June 24, 2007

Abstract

For graphs of bounded maximum degree, we consider acyclic t-improper colourings, that is, colourings in which each bipartite subgraph consisting of the edges between two colour classes is acyclic and each colour class induces a graph with maximum degree at most t.

We consider the supremum, over all graphs of maximum degree at most d, of the acyclic t-improper chromatic number and provide t-improper analogues of results by Alon, McDiarmid and Reed (1991, RSA 2(3), 277–288) and Fertin, Raspaud and Reed (2004, JGT 47(3), 163–182).

Submission date: 27 February 2007.

1 Introduction

Given a graph $G = (V, E)$, a proper colouring $\mathcal{V} = (V_1, \ldots, V_k)$ of V is *acyclic* if for all $1 \leq i < j \leq k$, the subgraph of G induced by $V_i \cup V_j$, which we denote $G[V_i \cup V_j]$, contains no cycles (i.e., is a forest). The *acyclic chromatic number* $\chi_a(G)$ is the smallest value k for which there exists a proper acyclic k-colouring of G. It is easily seen that $\chi_a(G) \leq \Delta(G)(\Delta(G) - 1) + 1$, as any proper colouring of the square G^2 of G is *de facto* a proper acyclic colouring of G, and G^2 has maximum degree at most $\Delta(G)(\Delta(G) - 1)$. In 1976, Erdős (see (cf. [1])) conjectured that $\chi_a(G) = o(\Delta(G)^2)$; this conjecture was proved

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[§]This author is supported by the Fonds québécois de la recherche sur la nature et les technologies.
[¶]Part of this work was supported by the European training network COMBSTRU (Combinatorial Structure of Intractable Problems), contract grant n° HPRN CT 2002 00278.
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by [2], who showed the existence of a fixed constant $c < 50$ such that for all G, $\chi_a(G) \leq c\Delta(G)^{4/3}$. [2] also showed that their bound was close to optimal by proving via probabilistic arguments that for Δ large,

$$\max\{\chi_a(G) : \Delta(G) \leq \Delta\} = \Omega\left(\frac{\Delta^{4/3}}{(\log \Delta)^{1/3}}\right).$$

When studying the asymptotics of $\chi_a(G)$ in terms of $\Delta(G)$, the restriction that the colouring be *proper* is not of great importance. Indeed, suppose we define the *relaxed acyclic chromatic number* $\chi_r(G)$ to be the smallest value k for which there exists a colouring $\mathcal{V} = (V_1, \ldots, V_k)$ of G such that, for all $1 \leq i < j \leq k$, $G[V_i \cup V_j]$ is a forest (placing no restriction on edges within a given block $G[V_i]$). Clearly, $\chi_r(G) \leq \chi_a(G)$. On the other hand, given such a colouring, it follows in particular that for all $1 \leq i \leq k$, $G[V_i]$ is a forest, so $\chi(G[V_i]) \leq 2$. By splitting V_i into stable sets $V_i^{(1)}$ and $V_i^{(2)}$ (for each $1 \leq i \leq k$), we may then obtain an acyclic *proper* colouring of G with at most $2k$ colours. It follows that $\chi_a(G)$ and $\chi_r(G)$ are within a factor of two of each other.

In this paper we investigate another relaxation of the acyclic chromatic number; in order to define it we first note that we may reformulate the definition of $\chi_a(G)$ by observing that if V_i and V_j are distinct stable sets in G, then $G[V_i \cup V_j]$ is exactly the bipartite graph $G[V_i, V_j]$ containing all edges with one endpoint in V_i and one endpoint in V_j. We may then equivalently define $\chi_a(G)$ as the smallest value k for which there exists a proper colouring $\mathcal{V} = (V_1, \ldots, V_k)$ of V such that for all $1 \leq i < j \leq k$, $G[V_i, V_j]$ is a forest (i.e. such that with this colouring, G contains no *alternating cycle*).

Starting from this definition, we may now relax the requirement that $\mathcal{V}$ be a proper colouring while continuing to impose that G contain no alternating cycle. To wit: given an integer $t \geq 0$, we say that a colouring $\mathcal{V} = (V_1, \ldots, V_k)$ is *t-improper* if for all $1 \leq i \leq k$, $G[V_i]$ has maximum degree at most t (in this case we say that V_i is *t-dependent*, for each $1 \leq i \leq t$). The *t-improper acyclic chromatic number* $\chi_a^t(G)$ is the smallest k for which there exists a t-improper colouring $\mathcal{V} = (V_1, \ldots, V_k)$ such that with this colouring, G contains no alternating cycle.

For an integer $d \geq 0$, we let

$$\chi_a^t(d) = \max\{\chi_t^t(G) : \Delta(G) \leq d\}.$$

The object of this paper is to study how $\chi_a^t(d)$ varies as a function of t and of d. Clearly, for any d, $\chi_a^0(d) \geq \chi_a^1(d) \geq \ldots \geq \chi_a^d(d) = 1$.

It is easily seen that $\chi_a^t(d) = \Omega\left((d/t)^{4/3}/(\ln d)^{1/3}\right)$; given an acyclic t-improper colouring, by applying the first of the results from [2] mentioned above, we can acyclically colour each colour class with at most $ct^{4/3}$ new colours (where c is some fixed constant which is less than 50) to obtain an acyclic colouring of the entire graph. Our first result is to show that this straightforward lower bound on $\chi_a^t(d)$ can be much improved upon asymptotically, as long as $t \leq d - 10\sqrt{d \ln d}$. More fully,

Theorem 1. *If $t \le d - 10\sqrt{d \ln d}$, then $\chi_a^t(d) = \Omega\left((d-t)^{4/3}/(\ln d)^{1/3}\right)$.*

In particular, if $t = (1-\varepsilon)d$ for any fixed constant ε, $0 < \varepsilon \le 1$, then we obtain the same asymptotic lower bound as Alon *et al.* Comparing this lower bound with the upper bound $\chi_a^t(d) = O(d^{4/3})$, we see the surprising fact that even allowing $t = \Omega(d)$ does not greatly reduce the number of colours needed for improper acyclic colourings of graphs with large maximum degree.

At some point, $\chi_a^t(d)$ must drop significantly as t increases, because $\chi_a^d(d) = 1$. Although we are unable to pin down the behaviour of $\chi_a^t(d)$ viewed as a function of t, we can improve upon the upper bound of Alon *et al.* when t is very close to d (more precisely, when $d - t = o(d^{1/3})$). We prove:

Theorem 2. $\chi_a^t(d) = O(d \ln d + (d-t)d)$.

As for lower bounds on $\chi_a^t(d)$ when $d - t = o(d)$, we first note that [3] showed $\chi_a^{d-2}(d) \ge 3$; we can straightforwardly generalise this result by showing that $\chi_a^t(d) \ge d - t + 1$. This is done as follows: if K_{d+1} is the complete graph on $d+1$ vertices, then $\chi_a^t(K_{d+1}) \ge d - t + 1$, since, in any acyclic t-improper colouring of K_{d+1}, at most one colour class has more than one vertex and no colour class has more than $t+1$ vertices. We can, however, improve upon this further and, in the final section, we exhibit a set of examples showing the following lower bound.

Theorem 3. $\chi_a^{d-1}(d) = \Omega(d^{2/3})$.

We would like to reduce the gaps between the lower and upper bounds on $\chi_a^t(d)$. For $t = d-1$, the problem is particularly tantalising, and, in this case, the lower bound of Theorem 3 and the upper bound of Theorem 2 differ by a factor of $d^{1/3} \ln d$. For this choice of t, the problem also includes the conjecture from [3] that every subcubic graph is acyclically 2-improperly 2-colourable.

In the rest of the paper, we use the following notation. The degree of a given vertex v is denoted by $d(v)$. A *k-vertex* (resp. a *$^{\le}k$-vertex*) is a vertex of degree k (resp. degree at most k). We denote by $N(v)$ the set of the neighbours of v. A k-cycle (resp. a *$^{\ge}k$-cycle*) is a cycle containing k vertices (resp. at least k vertices). For a graph G and a vertex $v \in V(G)$, we denote by $G \setminus \{v\}$ the graph obtained from G by removing v and its incident edges; for an edge uv of $E(G)$, $G \setminus \{uv\}$ denotes the graph obtained from G by removing the edge uv. These notions are extended to sets of vertices and edges in an obvious way. Let G be a graph and f be a colouring of G. For a given vertex v of G, we denote by $\mathrm{im}_f(v)$, or simply $\mathrm{im}(v)$ when the colouring is clear from the context, the number of neighbours of v having the same colour as v and call this quantity the *impropriety* of the vertex v. For notation not defined here, we refer the reader to [9].

2 A probabilistic lower bound for $\chi_a^t(d)$

In this section, we prove Proposition 6 below, a more explicit version of Theorem 1. Our argument mirrors that of Alon *et al.* but uses upper bounds on the t-

dependence number α^t, the size of a largest t-dependent set, in the random graph $G_{n,p}$. For more precise upper bounds on $\alpha^t(G_{n,p})$, consult [7].

Lemma 4. *Fix an integer $n \geq 1$ and $p \in \mathbb{R}$ with $4(\ln n/n)^{1/4} \leq p \leq 1$. Let $m = \lfloor n - 128 \ln n/p^4 \rfloor$. Then asymptotically almost surely and uniformly over p in the above range, any colouring of $G_{n,p}$ with $k \leq (n-m)/4$ colours and in which each colour class contains at most m vertices contains an alternating 4-cycle.*

Proof. As there are at most $k^n \leq n^n$ possible k-colourings of $G_{n,p}$, to prove the lemma it suffices to show that for any fixed k-colouring of the vertices of $G_{n,p}$ (which we denote $\{v_1, \ldots, v_n\}$) with colour classes $C_1, \ldots, C_k$ in which $|C_i| \leq m$ for all $1 \leq i \leq k$, the probability that $G_{n,p}$ does not contain an alternating 4-cycle is $o(n^{-n})$.

Fix a colouring as above, and let q be minimal such that $|C_1 \cup \ldots \cup C_q| \geq (n-m)/2$. Let $A = C_1 \cup \ldots \cup C_q$ and let $B = C_{q+1} \cup \ldots \cup C_k$. As no colour class has size greater than m, $|A| \leq (n+m)/2$ and so $|B| \geq (n-m)/2$. By symmetry, we may also assume that $|A| \geq n/2$.

Next, let $P = \{\{x_1, x'_1\}, \ldots, \{x_r, x'_r\}\}$ be a maximal collection of pairs of elements of A such that for $1 \leq i \leq r$, x_i and x'_i have the same colour, and for $1 \leq i < j \leq r$, $\{x_i, x'_i\}$ and $\{x_j, x'_j\}$ are disjoint. As we may place all but perhaps one vertex from each colour class C_i in some such pair (with one vertex left over precisely if $|C_i|$ is odd), it follows that

$$r \geq \frac{1}{2}(|A| - q) \geq \frac{1}{2}\left(\frac{n}{2} - k\right) \geq \frac{n}{8}.$$

Similarly, let $Q = \{\{y_1, y'_1\}, \ldots, \{y_s, y'_s\}\}$ be a maximal collection of pairs of elements of B satisfying identical conditions; by an identical argument to that above, it follows that $s \geq (n-m)/8$.

Let E be the event that for all $1 \leq i \leq r$, $1 \leq j \leq s$, $\{x_i, y_j, x'_i, y'_j\}$ is not an alternating 4-cycle, and let E' be the event that $G_{n,p}$ contains no alternating 4-cycle; clearly $E' \subseteq E$. For *fixed* $1 \leq i \leq r$ and $1 \leq j \leq s$, the probability that $\{x_i, y_j, x'_i, y'_j\}$ is not an alternating 4-cycle is $(1-p^4)$ and this event is independent from all other such events. As $(n-m) \geq 128 \ln n/p^4$ it follows that

$$\begin{aligned}\mathbf{Pr}(E') \leq \mathbf{Pr}(E) &\leq (1-p^4)^{rs} \leq e^{-p^4 rs} \\ &\leq \exp\left\{-\frac{p^4 n(n-m)}{64}\right\} \leq e^{-2n \ln n} = o(n^{-n}),\end{aligned}$$

as required. □

Using this lemma, we next bound the acyclic t-improper chromatic number of $G_{n,p}$ for p in the range allowed in Lemma 4.

Lemma 5. *Fix an integer $n \geq 1$ and $p \in \mathbb{R}$ with $4(\ln n/n)^{1/4} \leq p \leq 1$. Let $m = \lfloor n - 128 \ln n/p^4 \rfloor$ and let $t(n,p) = p(m-1) - 2\sqrt{np}$. Then asymptotically almost surely, for all integers $t \leq t(n,p)$, $\chi_a^t(G_{n,p}) \geq 32 \ln n/p^4$, uniformly over p and t in the above ranges.*

Proof. Fix n and p as above, and choose $t \leq t(n,p)$. We will show that asymptotically almost surely $G_{n,p}$ contains no t-dependent set of size greater than m, from which the claim follows immediately by applying Lemma 4 as $(n-m)/4 \geq 32 \ln n/p^4$. Let $G[m]$ represent the subgraph of $G_{n,p}$ induced by $\{v_1, \dots, v_m\}$. By a union bound and symmetry, we have

$$\mathbf{Pr}\left(\alpha^t(G_{n,p}) \geq m\right) \leq \binom{n}{m}\mathbf{Pr}\left(\Delta(G[m]) \leq t\right) \leq 2^n \mathbf{Pr}\left(\Delta(G[m]) \leq t\right).$$

Since, if $\Delta(G[m]) \leq t$ then $G[m]$ has at most $tm/2$ edges, it follows that

$$\begin{aligned}\mathbf{Pr}\left(\alpha^t(G_{n,p}) \geq m\right) &\leq 2^n \mathbf{Pr}\left(E(G[m]) \leq \frac{tm}{2}\right) \\ &\leq 2^n \mathbf{Pr}\left(E(G[m]) - p\binom{m}{2} \leq \frac{tm}{2} - p\binom{m}{2}\right)\end{aligned}$$

Finally, by a Chernoff bound and by the definition of $t(n,p)$, we conclude that

$$\begin{aligned}\mathbf{Pr}\left(\alpha^t(G_{n,p}) \geq m\right) &\leq 2^n \exp\left\{-\left(\frac{tm}{2} - p\binom{m}{2}\right)^2 \cdot \left(2p\binom{m}{2}\right)^{-1}\right\} \\ &\leq 2^n \exp\left\{-\frac{(t-p(m-1))^2}{4p}\right\} \leq (2/e)^n = o(1),\end{aligned}$$

as claimed. □

Using Lemma 5, it is a straightforward calculation to bound $\chi_a^t(d)$ for d sufficiently large and t sufficiently far from d.

Proposition 6. *For all sufficiently large integers d and all non-negative integers $t \leq d - 10\sqrt{d \ln d}$,*

$$\chi_a^t(d) \geq \frac{(d-t)^{4/3}}{2^{14}(\ln d)^{1/3}}.$$

Proof. Choose n so that

$$2^{13} n^3 \ln n \leq d^3(d-t) \leq 2^{14} n^3 \ln n; \tag{1}$$

such a choice of n clearly exists as long as d is large enough. Let $p = (d - 4\sqrt{d \ln d})/n$; we first check that p and t satisfy the requirements of Lemma 5. Presuming d is large enough that $np \geq d/2$, by the lower bound in (1) and the fact that $d(d-t) \leq d^2$ we have

$$p \geq \frac{d}{2n} \geq \frac{(d^3(d-t))^{1/4}}{2n} \geq \frac{8n^{3/4}(\ln n)^{1/4}}{2n} = 4\left(\frac{\ln n}{n}\right)^{1/4}. \tag{2}$$

Furthermore, letting $m = \lfloor n - 128 \ln n/p^4 \rfloor$, we have

$$\begin{aligned}p(m-1) - 2\sqrt{np} &\geq np - \frac{128 \ln n}{p^3} - 2\sqrt{np} - 2 = d - 4\sqrt{d \ln d} - 2\sqrt{np} - 2 - \frac{128 \ln n}{p^3} \\ &\geq d - 8\sqrt{d \ln d} - \frac{128 \ln n}{p^3}.\end{aligned} \tag{3}$$

Since $p \geq d/2n$ and by the lower bound in (1),

$$\frac{128 \ln n}{p^3} \leq \frac{2^{10} n^3 \ln n}{d^3} \leq \frac{d-t}{8},$$

which combined with (3) yields

$$\begin{aligned} p(m-1) - 2\sqrt{np} &> d - 8\sqrt{d \ln d} - \frac{(d-t)}{8} \\ &= t + \frac{7(d-t)}{8} - 8\sqrt{d \ln d} > t, \end{aligned} \tag{4}$$

the last inequality holding since $t \leq d - 10\sqrt{d \ln d}$. As (2) and (4) hold we may apply Lemma 5 to bound $\chi_a^t(G_{n,p})$ with this choice of t and p; as $n > d$, it follows that as long as d is sufficiently large,

$$\mathbf{Pr}\left(\chi_a^t(G_{n,p}) \geq \frac{32 \ln n}{p^4}\right) \geq \frac{3}{4}, \tag{5}$$

say. Furthermore, by a union bound and a Chernoff bound,

$$\begin{aligned} \mathbf{Pr}\left(\Delta(G_{n,p}) > d\right) &\leq n \mathbf{Pr}\left(\mathrm{BIN}\left(n, \frac{d - 4\sqrt{d \ln d}}{n}\right) > d\right) \\ &\leq n e^{-16 \ln d/3} \leq \frac{1}{n}, \end{aligned} \tag{6}$$

the last inequality holding as $\ln d \geq \ln n/2$ (which is an easy consequence of (1)). Combining (5) and (6), we obtain that

$$\mathbf{Pr}\left(\chi_a^t(G_{n,p}) \geq \frac{32 \ln n}{p^4}, \Delta(G_{n,p}) \leq d\right) \geq \frac{3}{4} - \frac{1}{n} \geq \frac{1}{2}$$

as long as $n \geq 4$, so there is some graph G with maximum degree at most d and with $\chi_a^t(G) \geq 32 \ln n/p^4$. Since χ_a^t is monotonically increasing in d, it follows that

$$\chi_a^t(d) \geq \frac{32 \ln n}{p^4} > \frac{32 n^4 \ln n}{d^4}. \tag{7}$$

An easy calculation using the upper bound in (1) and the fact that $\ln n < 2 \ln d$ gives the bound

$$d^4 < \frac{2^{19} n^4 (\ln d)^{4/3}}{(d-t)^{4/3}},$$

so $32 n^4 \ln n/d^4 > (d-t)^{4/3}/2^{14}(\ln d)^{1/3}$. By (7), it follows that

$$\chi_a^t(d) \geq \frac{(d-t)^{4/3}}{2^{14}(\ln d)^{1/3}},$$

as claimed. □

3 A probabilistic upper bound for $\chi_a^t(d)$

In this section, we study the situation when t is even closer to d, when $d-t = o(d^{1/2})$ in particular. Theorem 2 is a corollary of our main result here.

We analyse a different parameter from, but one that is closely related to, the acyclic t-improper chromatic number. A *star colouring* of G is a colouring such that no path of length three (i.e. with four vertices) is alternating; in other words, each bipartite subgraph consisting of the edges between two colour classes is a disjoint union of stars. The *star chromatic number* $\chi_s(G)$ is the least number of colours needed in a proper star colouring of G. We analogously define the parameters $\chi_s^t(G)$ and $\chi_s^t(d)$ in the natural way. The star chromatic number was one of the main motivations for the original study of acyclic colourings [6]. Clearly, any star colouring is acyclic; thus, $\chi_a^t(d) \le \chi_s^t(d)$. Fertin, Raspaud and Reed [5] showed that $\chi_s(d) = O(d^{3/2})$ and that $\chi_s(d) = \Omega\left(d^{3/2}/(\ln d)^{1/2}\right)$. We note that a natural adaptation to star colouring of the argument given in the last section gives the following:

Theorem 7. *There exists a fixed constant $C > 0$ such that, if $t \le d - C\sqrt{d \ln d}$, then $\chi_s^t(d) = \Omega\left((d-t)^{3/2}/(\ln d)^{1/2}\right)$.*

Given a graph G of maximum degree d, the idea behind our method for improved upper bounds is to find a dominating set $\mathcal{D}$ and a function $g = g(d) = o(d^{3/2})$ such that $\left|\left(N(v) \cup N^2(v)\right) \cap \mathcal{D}\right| \le g$ for all $v \in V(G)$. Given such a set $\mathcal{D}$ in G, we assign colours to the vertices in $\mathcal{D}$ by greedily colouring $\mathcal{D}$ in the square of G (i.e. vertices in $\mathcal{D}$ at distance at most two in G receive different colours) with at most $g+1$ colours; then we give the vertices of $G \setminus \mathcal{D}$ the colour $g+2$. It can be verified that this colouring prevents any alternating paths of length three (and so prevents alternating cycles) and ensures that every vertex has at least one neighbour of a different colour. Furthermore, we can generalise this idea by prescribing that our set $\mathcal{D}$ is *k-dominating* — each vertex outside of $\mathcal{D}$ has at least k neighbours in $\mathcal{D}$ — to give a bound on $\chi_s^{d-k}(d)$.

Theorem 8. *$\chi_s^t(d) = O(d \ln d + (d-t)d)$.*

This result provides an asymptotically better upper bound than $\chi_s^t(d) = O(d^{3/2})$ when $d - t = o(d^{1/2})$. It also provides a better bound than $\chi_a^t(d) = O(d^{4/3})$ when $d-t = o(d^{1/3})$. Theorem 8 is an easy consequence of the following lemma:

Lemma 9. *Given a d-regular graph G and an integer $k \ge 1$, let $\psi(G,k)$ be the least integer $k' \ge k$ such that there exists a k-dominating set $\mathcal{D}$ for which, for all $v \in V(G)$, $|N(v) \cap \mathcal{D}| \le k'$. Let $\psi(d,k)$ be the maximum over all d-regular graphs G of $\psi(G,k)$. Then, for all d sufficiently large, $\psi(d,k) \le \max\{3k, 31 \ln d\}$.*

We postpone the proof of this lemma, first using it to prove Theorem 8:

Proof of Theorem 8. We first remark that the function χ_s^t is monotonic with respect to graph inclusion in the following sense: if G and G' are graphs with $V(G) \subseteq V(G')$, $\Delta(G) = \Delta(G')$ and $E(G) \subset E(G')$, then $\chi_s^t(G) \le \chi_s^t(G')$. As any graph G of maximum degree d is a subgraph of a d-regular graph (possibly with a greater number of vertices), to prove that $\chi_s^t(d) = O(d \ln d + (d-t)d)$ it therefore suffices to show that $\chi_s^t(G) = O(d \ln d + (d-t)d)$ for d-regular graphs G. We hereafter assume G is d-regular and d is large enough to apply Lemma 9. Let $k = d - t$. We will show that $\chi_s^t(G) \le d\psi(d,k) + 2$, which proves the theorem.

By Lemma 9, there is a k-dominating set $\mathcal{D}$ such that $|N(v) \cap \mathcal{D}| \le \psi(d,k)$ for all $v \in V(G)$. Fix such a dominating set $\mathcal{D}$ and form the auxiliary graph H as follows: let H have vertex set $\mathcal{D}$ and let uv be an edge of H precisely if u and v have graph distance at most two in G. As $|N(v) \cap \mathcal{D}| \le \psi(d,k)$ for all $v \in V(G)$, H has maximum degree at most $d\psi(d,k)$.

To colour G, we first greedily colour H using at most $d\psi(d,k) + 1$ colours, and assign each vertex v of $\mathcal{D}$ the colour it received in H. We next choose a new colour not used on the vertices of $\mathcal{D}$, and assign this colour to all vertices of $V(G) \setminus \mathcal{D}$. We remind the reader that $\mathrm{im}(v)$ denotes the number of neighbours of v of the same colour as v. If $v \in \mathcal{D}$ then $\mathrm{im}(v) = 0$, and if $v \in V \setminus \mathcal{D}$ then $\mathrm{im}(v) \le d - |N(v) \cap \mathcal{D}| \le d - k = t$, so the resulting colouring is t-improper.

Furthermore, given any path $P = v_1v_2v_3v_4$ of length three in G, either two consecutive vertices v_i, v_{i+1} of P are not in $\mathcal{D}$ (in which case $c(v_i) = c(v_{i+1})$ and P is not alternating), or two vertices v_i, v_{i+2} are in $\mathcal{D}$ (in which case $c(v_i) \ne c(v_{i+2})$ and P is not alternating). Thus, the above colouring is a star colouring G of impropriety at most t and using at most $d(3k + 31 \ln d) + 2$ colours; as G was an arbitrary d-regular graph, it follows that $\chi_s^t(d) \le d\psi(d,k) + 2$, as claimed. □

We next prove Lemma 9 with the aid of the following symmetric version of the Lovász Local Lemma:

Lemma 10 ([4], cf. [8], page 40)**.** *Let $\mathcal{A}$ be a set of bad events such that for each $A \in \mathcal{A}$*

1. $\mathbf{Pr}(A) \le p < 1$*, and*

2. *A is mutually independent of a set of all but at most δ of the other events.*

If $4p\delta \le 1$, then with positive probability, none of the events in $\mathcal{A}$ occur.

Proof of Lemma 9. We may clearly assume that k is at least $(31/3)\ln d$, since, if the claim of the lemma holds for such k, then it also holds for smaller k. Let $p = 2k/d$ and let $\mathcal{D}$ be a random set obtained by independently choosing each vertex v with probability p. We claim that, with positive probability, $\mathcal{D}$ is a k-dominating set such that $|N(v) \cap \mathcal{D}| \le 3k$ for all $v \in V(G)$; we will prove our claim using the local lemma.

For $v \in V(G)$, let A_v be the event that either $|N(v) \cap \mathcal{D}| < k$ or $|N(v) \cap \mathcal{D}| > 3k$. By the mutual independence principle, cf. [8], page 41, A_v is mutually independent of all but at most d^2 events A_w (with $w \ne v$). Furthermore, since $|N(v) \cap \mathcal{D}|$ has a binomial distribution with parameters d and p, we have by a Chernoff bound that

$$\mathbf{Pr}(A_v) = \mathbf{Pr}\left(||N(v) \cap \mathcal{D}| - \mathbf{E}(|N(v) \cap \mathcal{D}|)| > k\right) \le 2e^{-k/5} = o(d^{-2})$$

so $4\mathbf{Pr}(A_v)\,d^2 < 1$ for d large enough. By applying Lemma 10 with $\mathcal{A} = \{A_v \mid v \in V\}$, it follows that with positive probability none of the events A_v occur, i.e. $\mathcal{D}$ has the desired properties. □

4 A deterministic lower bound for $\chi_a^{d-1}(d)$

In this section, we concentrate on the case $t = d-1$ and exhibit an example G_n which gives the asymptotic lower bound of Theorem 3. Given a positive integer n, we construct the graph G_n as follows: G_n has vertex set $\{v_{ij} : i, j \in \{1, \dots n\}\} \cup \{w_{ij} : i, j \in \{1, \dots, n\}\}$. For $i, j \in \{1, \dots, n\}$ we let $\mathcal{V}_{ij} = \{v_{ij}, w_{ij}\}$. We can think of the set of vertices as an n-by-n matrix, each entry of which has been "doubled". Within each column $\mathcal{C}_i = \bigcup_{j=1}^n \mathcal{V}_{ij}$ and within each row $\mathcal{R}_j = \bigcup_{i=1}^n \mathcal{V}_{ij}$ we add all possible edges. The graph G_n has $2n^2$ vertices and is regular with degree $d = 4n - 3$. We will prove the following proposition, which directly implies Theorem 3:

Proposition 11. $\chi_a^{d-1}(G_n) \geq \frac{n}{n^{1/3}+1} + 1$.

Proof. Let $f : G_n \to \{1, \dots, k\}$ be an acyclic $(d-1)$-improper colouring of G_n; we will show that necessarily $k \geq \frac{n}{n^{1/3}+1}$. Since $n \geq 1$ it follows that $n/2 \geq \frac{n}{n^{1/3}+1}$ and thus we may assume that $k < n/2$. Clearly, some colour – say a_1 – appears on two vertices x, x' of $\mathcal{C}_1$. We call the colour a_1 "black" and refer to vertices receiving colour a_1 as black vertices. If $y, y' \in \mathcal{C}_1$ both receive colour $i \neq a_1$, then $xyx'y'$ forms an alternating cycle, so a_1 is the only colour appearing twice in $\mathcal{C}_1$. It follows that at most $k-1$ vertices in $\mathcal{C}_1$ are not black.

Applying the same logic to any column $\mathcal{C}_i$, we see that all but $k-1$ vertices in $\mathcal{C}_i$ must receive the same colour, say a_i. Since $k < n/2$, it is easily seen, then, that there must be a row $\mathcal{R}_m$ such that v_{m1} and w_{m1} are both black, and v_{mi} and w_{mi} are both coloured a_i. This implies that $a_i = a_1$, since otherwise $v_{m1}v_{mi}w_{m1}w_{mj}$ would be an alternating cycle. It follows that in all columns, at most $k-1$ vertices receive a colour other than a_1. Symmetrically, there is a colour b such that in all rows, at most $k-1$ vertices receive a colour other than b; clearly, it must the case that $b = a_1$.

If there are $i, j \in \{1, \dots, n\}$ such that both $\mathcal{R}_i$ and $\mathcal{C}_j$ are entirely coloured black, then all the neighbours of v_{ij}, w_{ij} are coloured with a_1 and the colouring is not $(d-1)$-improper; therefore, it must be the case that either all rows, or all columns, contain a non-black vertex. Without loss of generality, we may assume that all rows contain a non-black vertex.

Let $x_1, \dots, x_r$ be non-black vertices receiving the same colour, say a, and let $x_i \in \mathcal{V}_{\ell_i, m_i}$, for $1 \leq i \leq r$. As previously noted, no two of $x_1, \dots, x_r$ may lie in the same row or column; i.e., for $i \neq j$, $\ell_i \neq \ell_j$ and $m_i \neq m_j$.

Claim 1. *At least $3\binom{r}{2}$ vertices of $\bigcup_{1 \leq i \neq j \leq r} \mathcal{V}_{\ell_i, m_j}$ receive a non-black colour other than a.*

Proof. No vertices in $\bigcup_{1 \leq i \neq j \leq r} \mathcal{V}_{\ell_i, m_j}$ receive colour a as each such vertex is in the same row as one of $x_1, \dots, x_r$. On the other hand, for each pair i, j with

$1 \leq i < j \leq r$, at least three of the vertices in $\mathcal{V}_{\ell_i,m_j} \cup \mathcal{V}_{\ell_j,m_i}$ must receive a colour other than a_1. For if $y, y' \in \mathcal{V}_{\ell_i,m_j} \cup \mathcal{V}_{\ell_j,m_i}$ both receive colour a_1, then $x_i y x_j y'$ forms an alternating cycle. The result follows as there are $\binom{r}{2}$ pairs i, j with $1 \leq i < j \leq r$. □

Claim 2. *At least r distinct non-black colours appear on $\bigcup_{1\leq i<j\leq r} \mathcal{V}_{\ell_i,m_j}$.*

Proof. By an argument just as above, each of $\mathcal{V}_{\ell_1,m_2}, \ldots, \mathcal{V}_{\ell_1,m_r}$ must contain a vertex receiving a colour other than a_1 or a. These colours must all be distinct as $\mathcal{V}_{\ell_1,m_2}, \ldots, \mathcal{V}_{\ell_1,m_r}$ are all contained within $\mathcal{R}_{\ell_1}$. □

Let $\{a_2, a_3, \ldots, a_k\}$ be the set of non-black colours. Let $x_1^2, \ldots, x_{r_2}^2$ be the vertices receiving colour a_2, and for $i = 3, \ldots, k$ let $x_1^i, \ldots, x_{r_i}^i$ be the vertices receiving colour a_i *which are in a different row from all vertices in* $\bigcup_{j<i} \bigcup_{s\leq r_j} x_s^j$. As every row contains a non-black vertex, $\sum_{i=2}^k r_i = n$; it is possible that $r_i = 0$ for certain i, if there is a vertex coloured with one of $a_2, \ldots, a_i$ in every row.

For $i \in \{2, \ldots, k\}$ and $s \in \{1, \ldots, r_i\}$, say vertex $x_s^i \in \mathcal{V}_{\ell_s^i,m_s^i}$, and let

$$A_i = \bigcup_{1\leq s<t\leq r_i} \mathcal{V}_{\ell_s^i,m_t^i} \cup \mathcal{V}_{\ell_t^i,m_s^i}.$$

By Claim 1, at least $3\binom{r_i}{2}$ vertices of A_i are non-black. Furthermore, if $i \neq i'$ then for any $s \in \{1, \ldots, r_i\}$, $s' \in \{1, \ldots, r_{i'}\}$, x_s^i and $x_{s'}^{i'}$ are in different rows – so A_i and $A_{i'}$ are disjoint. It follows that in $\bigcup_{i=2}^k A_i \cup \{x_1^i, \ldots, x_{r_i}^i\}$, at least

$$\sum_{i=2}^{k} \left(3\binom{r_i}{2} + r_i\right) \geq \sum_{i=2}^{k} r_i^2 \tag{8}$$

vertices are non-black. As $\sum_{i=2}^k r_i = n$, it is easily seen that

$$\sum_{i=2}^{k} r_i^2 \geq (k-1)\left(\left\lfloor \frac{n}{k-1} \right\rfloor\right)^2.$$

As there are only $k-1$ non-black colours, it follows that some non-black colour – say a_2 – appears at least $(\lfloor n/(k-1) \rfloor)^2$ times. If $(\lfloor n/(k-1) \rfloor)^2 \geq n^{2/3}$, then by Claim 2, at least $n^{2/3} + 1 > \frac{n}{n^{1/3}+1} + 1$ colours appear on G_n, so we may assume that $n^{2/3} > (\lfloor n/(k-1) \rfloor)^2 \geq (n/(k-1) - 1)^2$. But then $k > \frac{n}{n^{1/3}+1} + 1$, as claimed. □

Since $d = 4n-3$, the above proposition yields $\chi_a^{d-1}(G_n) \geq (1+o(1))2^{-4/3} d^{2/3}$. It is worth noting that the correct asymptotic order of $\chi_a^{d-1}(G_n)$ is unknown; it is even conceivable that $\chi_a^{d-1}(G_n) = \Theta(d)$. For improper star colouring, a construction and accompanying argument that are similar to the above gives $\chi_s^{d-1}(d) \geq (1 + o(1))2^{-1/6} d^{2/3}$.

5 Conclusion

In our view, the most surprising result of this paper is that the same asymptotic lower bound for ordinary acyclic chromatic number by Alon *et al.* also holds for the acyclic t-improper chromatic number for any $t = t(d)$ satisfying $d-t = \Theta(d)$. As $\chi_a(G) \geq \chi_a^t(G)$ for any $t \geq 0$, this means that, for $d - t = \Theta(d)$, Theorem 1 is asymptotically tight up to a factor of $(\ln d)^{1/3}$.

In the case that t is very close to d, Theorem 8 improves upon upper bounds for $\chi_a^t(d)$ and $\chi_s^t(d)$ implied by the results of Alon *et al.* and Fertin *et al.*, respectively, giving for instance that $\chi_s^t(d) = O(d \ln d)$ for $d-t = O(\ln d)$. On the other hand, we showed that $\chi_a^{d-1}(d) = \Omega(d^{2/3})$ by a deterministic construction.

<table>
<tr><th rowspan="2">$d-t$</th><th colspan="2">$\chi_a^t(d)$</th><th colspan="2">$\chi_s^t(d)$</th></tr>
<tr><th>lower</th><th>upper</th><th>lower</th><th>upper</th></tr>
<tr><td>$\Theta(d)$</td><td>$\Omega\left(\frac{d^{4/3}}{(\ln d)^{1/3}}\right)$</td><td rowspan="3">$O(d^{4/3})$</td><td>$\Omega\left(\frac{d^{3/2}}{(\ln d)^{1/2}}\right)$</td><td rowspan="2">$O(d^{3/2})$</td></tr>
<tr><td>$\omega(\sqrt{d \ln d})$</td><td>$\Omega\left(\frac{(d-t)^{4/3}}{(\ln d)^{1/3}}\right)$</td><td>$\Omega\left(\frac{(d-t)^{3/2}}{(\ln d)^{1/2}}\right)$</td></tr>
<tr><td>$O(d^{1/2})$</td><td rowspan="3">$\Omega\left(d^{2/3}\right)$</td><td rowspan="3">$\Omega\left(d^{2/3}\right)$</td><td rowspan="2">$O((d-t)d)$</td></tr>
<tr><td>$O(d^{1/3})$</td><td>$O((d-t)d)$</td></tr>
<tr><td>$O(\ln d)$</td><td>$O(d \ln d)$</td><td>$O(d \ln d)$</td></tr>
<tr><td>0</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Table 1: Asymptotic bounds for $\chi_a^t(d)$ and $\chi_s^t(d)$.

There is much remaining work in the case $d - t = o(d)$. Table 1 is a rough summary of the current bounds on $\chi_a^t(d)$ and $\chi_s^t(d)$ when d is large. A case of particular interest to the authors is when $d-t = 1$; in this case, it is unknown if $\chi_a^{d-1}(d)$ is $\Theta(d^{2/3})$, $\Theta(d \ln d)$ or lies somewhere strictly between these extremes.

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