

BARYCENTRIC BROWNIAN BEES

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Abstract. We establish an invariance principle for the barycenter of a Brunet-Derrida particle system in d dimensions. The model consists of N particles undergoing dyadic branching Brownian motion with rate 1. At a branching event, the number of particles is kept equal to N by removing the particle located furthest away from the barycenter. To prove the invariance principle, a key step is to establish Harris recurrence for the process viewed from its barycenter.

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1. Introduction

The *barycentric Brownian bees* processes are Brunet–Derrida particle systems defined as follows. The population consists of N individual particles, whose positions are points in \mathbb{R}^d . Independently, each particle moves according to a standard Brownian motion, and undergoes binary branching at rate one. The instant when a branching event occurs, the particle furthest from the current barycenter of the particles is removed. Our notation for such a process is

$$X = (X(t), t \geq 0) = ((X_i(t))_{i \in [N]}, t \geq 0).$$

We write $\bar{X}(t) = N^{-1} \sum_{1 \leq i \leq N} X_i(t)$ and $\bar{X} = (\bar{X}(t), t \geq 0)$, and call $\bar{X}(t)$ the *barycenter* of $X(t)$.

The result of the current work is an invariance principle for the barycenter process \bar{X} .

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Theorem 1.1. *For all $d \geq 1$ and $N \geq 1$, there exists $\sigma = \sigma(d, N) \in (0, \infty)$ such that, as $m \rightarrow \infty$,*

$$\left(m^{-1/2}\overline{X}(tm), 0 \leq t \leq 1\right) \xrightarrow{d} (\sigma B(t), 0 \leq t \leq 1),$$

with respect to the Skorohod topology on $\mathcal{D}([0, 1], \mathbb{R}^d)$, where $(B(t), 0 \leq t \leq 1)$ is a standard Brownian motion in \mathbb{R}^d starting at the origin.

It turns out that $\sigma(d, 1) = 1 = \sigma(d, 2)$ for all $d \geq 1$. For $N \geq 3$, our proof does not yield insight into the value of $\sigma(d, N)$. We point out that whatever the initial configuration $X(0)$, it is sent to the origin by this scaling. Since the Skorohod topology relativized to $C([0, 1], \mathbb{R}^d)$ coincides with the uniform topology, it is not hard to see that any reasonable smoothing of $(m^{-1/2}\overline{X}(tm), 0 \leq t \leq 1)$ will converge in distribution to $(\sigma B(t), 0 \leq t \leq 1)$ in the uniform topology on $C([0, 1], \mathbb{R}^d)$. For example, one may linearly interpolate $(m^{-1/2}\overline{X}(tm), 0 \leq t \leq 1)$ at integer times.

A key step in proving Theorem 1.1 is to show that, when viewed from its barycenter, the process is Harris recurrent.

Definition 1.2. (Harris recurrence, [14]) *We say that a time-homogeneous càdlàg Markov process $\Phi = (\Phi_t, t \geq 0)$ on the state space $\mathbb{R}^{d \times N}$ is Harris recurrent if there exists a σ -finite Borel measure φ on $\mathbb{R}^{d \times N}$ such that for any Borel set $A \subset \mathbb{R}^{d \times N}$ with $\varphi(A) > 0$, for all $x \in \mathbb{R}^{d \times N}$,*

$$\mathbf{P}_x(\eta_A = \infty) = 1$$

where $\eta_A := \int_0^\infty \mathbb{1}_{\{\Phi_t \in A\}} dt$ is the total time spent in A by Φ .

Throughout the paper, we will denote particle configurations by $x = (x_i)_{i \in [N]} \in \mathbb{R}^{d \times N}$ where each $x_i \in \mathbb{R}^d$. Given a configuration $x \in \mathbb{R}^{d \times N}$, and another point $y \in \mathbb{R}^d$, we write $x - y := (x_i - y)_{i \in [N]}$. Equipped with this definition and notation, we state the second main result of the present work.

Theorem 1.3. *The process $X - \overline{X} := (X(t) - \overline{X}(t), t \geq 0)$ is Harris recurrent.*

1.1. Two constructions of barycentric Brownian bees, and a little additional notation

We work on an abstract probability space $(\Omega, \mathcal{F}, \mathbf{P})$, rich enough to support all of the random variables encountered.

There are two distinct constructions of the barycentric Brownian bees (hereafter BBB) process that will be useful at different points in our analysis. The first construction closely parallels that given in Section 2 of [7].

- Let $\mathcal{J} = (\mathcal{J}_t, t \geq 0)$ be the counting process for a Poisson point process on $[0, \infty)$ with rate N , and for $i \geq 0$ let $J_i = \inf\{t \geq 0 : \mathcal{J}_t = i\}$. The process \mathcal{J} will be the number of branching events of X up to time t , and $(J_i, i \geq 1)$ will be its branching times (and $J_0 = 0$).
- Let $(\mathcal{U}_i; i \geq 1)$ be independent random variables, uniformly distributed on $\{1, \dots, N\}$ and independent of \mathcal{J} .
- Let $(B_i)_{i \in [N]}$ be independent d -dimensional Brownian motions starting from the origin, which are independent of \mathcal{J} and of $(\mathcal{U}_i; i \geq 1)$.

With these definitions at hand, the picture to have in mind when reading the formal construction is this: at time J_i , particle \mathcal{U}_i branches, and some particle is killed. If the killed particle has index k , then we may equivalently view this as the particle with index k jumping to the location of the branching event, which is $X_{\mathcal{U}_i}(J_i-)$.

Formally, fix an initial configuration $x = (x_j)_{j \in [N]} \in \mathbb{R}^{d \times N}$. We may define the BBB inductively, as follows. Set $X_j(0) = x_j$ for each $j \in [N]$. Then, for each $i \geq 1$ and $j \in [N]$, let

$$X_j(t) := X_j(J_{i-1}) + B_j(t) - B_j(J_{i-1}), \quad \text{for } t \in [J_{i-1}, J_i).$$

This essentially states that in between jump times J_{i-1} and J_i , the BBB process is nothing more than N independent Brownian motions, appropriately shifted. Finally, for each $i \geq 1$, at time J_i- ,

we let particle \mathcal{U}_i branch, and kill (remove) the particle with index

$$k := \operatorname{argmax}_{j \in [N]} \left| X_j(J_i-) - \frac{1}{N+1} \left(X_{\mathcal{U}_i}(J_i-) + \sum_{l \in [N]} X_l(J_i-) \right) \right|.$$

In words, we kill the particle which is furthest from the barycenter; the barycenter is calculated taking the newly born particle into account. We do not need to worry about ties since they occur with probability 0. This means that at time J_i , we set

$$X_j(J_i) = \begin{cases} X_{\mathcal{U}_i}(J_i-) & j = k, \\ X_j(J_i-) & j \in [N] \setminus \{k\}. \end{cases} \quad (1.1)$$

This completes the first construction.

The second construction realizes the BBB process as embedded within a standard d -dimensional dyadic branching Brownian motion $W(t) = (W_i(t))_{i \in [\mathcal{N}(t)]}$. The function $\mathcal{N} : [0, \infty) \rightarrow \mathbb{N}$ counts the number of particles in the branching Brownian motion W at any given time. Our convention is that when a branching event occurs in W , the new particle is appended to the end of the list of existing particles; if the particle with index i branches at time τ then $\mathcal{N}(\tau) = \mathcal{N}(\tau-) + 1$ and

$$W_j(\tau) = \begin{cases} W_j(\tau-) & \text{if } j < \mathcal{N}(\tau), \\ W_i(\tau-) & \text{if } j = \mathcal{N}(\tau). \end{cases}$$

The branching property means that after time τ , the particles with indices i and $\mathcal{N}(\tau)$ evolve independently.

Given $x \in \mathbb{R}^{d \times N}$, we initialize W with N particles in \mathbb{R}^d at positions $(x_i)_{i \in [N]}$, so that $\mathcal{N}(0) = N$ and $W(0) = x$. To describe the embedding of X within W , it suffices to explain the evolution of the vector $I(t) = (I_1(t), \dots, I_N(t)) \in \mathcal{N}(t)^{[N]}$ of indices of particles belonging to the BBB process at all times $t \geq 0$.

First, $I(0) = (1, 2, \dots, N)$, and I is constant between branching events of W . If a particle with index $i \notin \{I_1(\tau-), \dots, I_N(\tau-)\}$ branches at time $\tau > 0$ then $I(\tau) = I(\tau-)$. Finally, if for some $i \in [N]$ the particle with index $I_i(\tau-)$ branches at time τ , then let

$$k = \operatorname{argmax}_{j \in [N]} \left| W_{I_j(\tau-)}(\tau) - \frac{1}{N+1} \left(W_{I_i(\tau-)}(\tau) + \sum_{l \in [N]} W_{I_l(\tau-)}(\tau) \right) \right|.$$

In other words, k is the index of the particle furthest from the barycenter, when the barycenter is calculated taking the newly born particle (which has index $\mathcal{N}(\tau)$ and position $W_{I_i(\tau-)}(\tau) = W_{\mathcal{N}(\tau)}(\tau)$) into account.

Now set

$$I_j(\tau) = \begin{cases} \mathcal{N}(\tau) & \text{if } j = k \\ I_j(\tau-) & \text{if } j \neq k. \end{cases}$$

We can then realize the process X as

$$X = (X_j(t))_{j \in [N]} := (W_{I_j(t)}(t))_{j \in [N]}. \quad (1.2)$$

The realization of the BBB process within a branching Brownian motion will be useful in particular in Section 2.

We note that the labelling conventions of particles are consistent between the two constructions, in the sense that from a given configuration $(X_j(t-))_{j \in [N]}$, if there is a branching event at time t , then the definitions of the vectors $(X_j(t))_{t \in [N]}$ given by (1.1) and by (1.2) agree with one other.

At one point in the paper it will be useful to allow the BBB process to start from a configuration with fewer than N particles. In this case no killing occurs until the population size reaches N ; a branching event which occurs before this time simply increases the population size by one (with the convention that the newly-born particle is appended to the end of the list of existing particles,

like in a BBM). For a BBB started from fewer than N particles, we write $N(t)$ for the number of particles at time t , so $X(t) = (X_i(t))_{i \in [N(t)]}$ and a.s. $N(t) = N$ for all t sufficiently large.

We write $(\mathcal{F}_t, t \geq 0)$ for the natural filtration of X ,

$$\mathcal{F}_t = \mathcal{F}_t(X) := \sigma((X_j(s))_{j \in [N]}; 0 \leq s \leq t).$$

We write $\mathbf{P}_x(\cdot) := \mathbf{P}(\cdot | X(0) = x)$ for $x \in \mathbb{R}^{d \times N}$. Likewise, for a Borel probability measure μ on $\mathbb{R}^{d \times N}$, we write \mathbf{P}_μ for the probability measure under which the process starts from a μ -distributed initial configuration, namely $\mathbf{P}_\mu(X(0) \in A) = \mu(A)$ for Borel $A \subset \mathbb{R}^{d \times N}$.

We use the notation $B(y, r)$ to denote the Euclidean ball centered at y of radius r in \mathbb{R}^d . Given two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ with $g \neq 0$, we write $f \stackrel{m \rightarrow \infty}{\sim} g$ if $\frac{f(m)}{g(m)} = 1 + o(1)$ as $m \rightarrow \infty$. Finally, for \mathbb{R} -valued random variables Z_1 and Z_2 , we say that Z_1 stochastically dominates Z_2 (with respect to a given probability measure \mathbf{P}), if

$$\mathbf{P}(Z_1 \geq y) \geq \mathbf{P}(Z_2 \geq y), \quad \forall y \in \mathbb{R}.$$

1.2. Overview of the proof

In this section, we first describe a general mechanism for obtaining invariance principles for càdlàg processes. We then sketch how it is applied in the current setting, and in particular how Harris recurrence comes into play, and conclude with an overview of the proof that, when viewed from its barycenter, the BBB process is Harris recurrent.

Fix $N, d \in \mathbb{N}$. Given an arbitrary $\mathcal{D}([0, \infty), \mathbb{R}^{d \times N})$ -valued stochastic process $X = ((X_j(t))_{j \in [N]}, t \geq 0)$, we define the barycenter $\bar{X}(t) := N^{-1} \sum_{1 \leq i \leq N} X_i(t)$; so $\bar{X} = (\bar{X}(t), t \geq 0)$ is an \mathbb{R}^d -valued stochastic process. We say that \bar{X} satisfies an invariance principle if there exist $\alpha \in \mathbb{R}^d$ and a $d \times d$ matrix $\Sigma = \Sigma(d, N)$ such that

$$\left(m^{-1/2} (\bar{X}(tm) - tm\alpha), 0 \leq t \leq 1 \right) \xrightarrow{d} (\Sigma B(t), 0 \leq t \leq 1) \quad (1.3)$$

with respect to the Skorohod topology on $\mathcal{D}([0, 1], \mathbb{R}^{d \times N})$, where $B = (B(t), 0 \leq t \leq 1)$ is a standard d -dimensional Brownian motion. The following general proposition identifies sufficient conditions for \bar{X} to satisfy an invariance principle.

Proposition 1.4. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ denote a probability space and $X : \Omega \rightarrow \mathcal{D}([0, \infty), \mathbb{R}^{d \times N})$ a càdlàg stochastic process with $X = ((X_j(t))_{j \in [N]}, t \geq 0)$ with $X(0) = x$ for some $x \in \mathbb{R}^{d \times N}$. Suppose that there exists a nondecreasing sequence $(\tau_i; i \geq 1)$ of \mathbf{P}_x -a.s. finite $[0, \infty)$ -valued random variables, and such that the following properties are satisfied.*

(1) *(IID time increments) The time increments*

$$(\tau_{i+1} - \tau_i; i \geq 1)$$

are independent and identically distributed, with finite mean.

(2) *(IID increments) The \mathbb{R}^d -valued random variables*

$$((X_1(\tau_{i+1}) - X_1(\tau_i)); i \geq 1)$$

are independent and identically distributed.

(3) *(Identical distribution between times) The \mathbb{R} -valued random variables*

$$\left(\sup_{\tau_i \leq t \leq \tau_{i+1}} |X_1(t) - X_1(\tau_i)|; i \geq 1 \right)$$

are identically distributed.

(4) *(Finite variance increments) We have*

$$\mathbf{E}_x \left[\sup_{\tau_1 \leq t \leq \tau_2} |X_1(t) - X_1(\tau_1)|^2 \right] < \infty.$$

(5) (*Barycenter approximation in probability*) We have

$$m^{-1/2} \sup_{0 \leq t \leq 1} |\bar{X}(tm) - X_1(tm)| \rightarrow 0$$

in probability as $m \rightarrow \infty$.

Then the barycenter \bar{X} of X satisfies an invariance principle (1.3). Moreover, the rescaling Σ and the drift α are given by

$$\Sigma = \mathbf{E}_x[\tau_2 - \tau_1]^{-1/2} Q$$

where Q is a d -by- d matrix such that $C := QQ^T$ is the covariance matrix of $X_1(\tau_2) - X_1(\tau_1)$, and

$$\alpha = \mathbf{E}_x[X_1(\tau_2) - X_1(\tau_1)] \mathbf{E}_x[\tau_2 - \tau_1]^{-1}.$$

The proof of Proposition 1.4 appears in the Appendix. To apply the proposition in our setting, we begin by constructing a sequence of “regeneration” times for the BBB process X with properties (1), (2) and (3) of Proposition 1.4 in mind. The prototypical regeneration time will be defined as the first time $\rho > 0$ when two specific events have occurred. The events are constructed in such a way that when they both occur, the process starts over (regenerates) from a single *queen particle*, by which we mean that all particles in the BBB at time ρ are descendants of $X_1(\rho - 1)$. At a time ρ when these two events have just taken place, the law of $(X_j(\rho) - X_1(\rho - 1))_{j \in [N]}$ is given by an explicit measure and is independent of the state of the process at any previous regeneration times $\rho' \leq \rho$.

In order to verify the conditions of Proposition 1.4, the key will be to control the amount of time required for regeneration from a single queen particle to occur. The events required for regeneration are defined in Section 2.1; for now we simply note that they shall involve (a) a specific branching pattern occurring over a bounded time interval, and (b) the particles staying in certain bounded regions over the same time interval. The probability that these events occur will be easy to bound from below, provided that all particles start at a bounded distance from their barycenter.

The latter requirement is closely connected to the Harris recurrence of $X - \bar{X}$. Harris recurrence essentially requires that a process eventually spends an infinite amount of time in any sufficiently large set, where “large” is measured by a σ -finite Borel measure which we are free to choose. We prove this for $X - \bar{X}$ by showing that the time required for X to reach a configuration in which all particles are a bounded distance from the barycenter has exponential tails (see Remark 2.6 (ii) below), so in particular has finite moments of all orders. This will easily yield Harris recurrence. It will also allow us to deduce that our renewal time increments have exponential tails, and thence to show that the renewal times satisfy the conditions of Proposition 1.4. Theorem 1.1 will then straightforwardly follow from Proposition 1.4.

The remainder of this section provides an overview of the proof that $\Phi := X - \bar{X}$ is Harris recurrent. In broad strokes, the proof proceeds as follows. Given given points x_1, \dots, x_N in \mathbb{R}^d , say that $x = (x_1, \dots, x_N)$ is *unambiguous* if for any vector (f_1, \dots, f_N) of non-negative integers with $\sum_{i=1}^N f_i = N + 1$,

$$\left| x_j - \frac{1}{N+1} \sum_{i=1}^N f_i x_i \right| \neq \left| x_k - \frac{1}{N+1} \sum_{i=1}^N f_i x_i \right| \quad (1.4)$$

for all $1 \leq j < k \leq N$. Here is the meaning of this condition. Suppose that at some point in the BBB process, the particles have positions x_1, \dots, x_N . Now imagine that a sequence of branching events occurs in rapid succession, rapidly enough that the particles essentially do not move during the course of the branching. If at some point during such a sequence, the moment a branching event occurs, there are f_i particles at (or extremely near to) position x_i (for each $1 \leq i \leq N$), then the barycenter of the configuration is at (or extremely close to)

$$\frac{1}{N+1} \sum_{i=1}^N f_i x_i.$$

If x is unambiguous, then at this point there is a unique particle which is furthest from the barycenter, so there is no ambiguity about which particle to kill.

We actually use a more quantitative version of the above definition. For $\delta > 0$, say that $x = (x_1, \dots, x_N)$ is δ -unambiguous if for any vector (f_1, \dots, f_N) of non-negative integers with $\sum_{i=1}^N f_i = N + 1$, for all $1 \leq j < k \leq n$, we have

$$\left| \left| x_j - \frac{1}{N+1} \sum_{i=1}^N f_i x_i \right| - \left| x_k - \frac{1}{N+1} \sum_{i=1}^N f_i x_i \right| \right| > \delta. \quad (1.5)$$

It is not hard to show that if $x = (x_1, \dots, x_N)$ is δ -unambiguous and $y = (y_1, \dots, y_N)$ is such that $|x_i - y_i| < \delta/4$ for $1 \leq i \leq N$, then for any non-negative integers (f_1, \dots, f_N) with $\sum_{i=1}^N f_i = N + 1$, the index j for which $\left| x_j - (N+1)^{-1} \sum_{i=1}^N f_i x_i \right|$ is maximized, also maximizes $\left| y_j - (N+1)^{-1} \sum_{i=1}^N f_i y_i \right|$. This means that in the BBB process, if particles stay within distance $\delta/4$ of a fixed δ -unambiguous configuration x for a given period of time, then during that time, for the purpose of identifying which particle is furthest from the barycenter, one may pretend that all particles within distance $\delta/4$ of point x_i are in fact colocated at x_i . In other words, during that time the effect of spatial motion can be ignored when calculating which particles are killed after branching events; we may just pretend all particles are located at one of x_1, \dots, x_N .

The following straightforward lemma implies that from any initial configuration, the BBB process quickly reaches a δ -unambiguous configuration for some $\delta > 0$. Write

$$\xi_\delta := \inf \{ t \geq 0 : X(t) \text{ is } \delta\text{-unambiguous} \}.$$

Lemma 1.5. *For any $\epsilon > 0$, there exists $\delta = \delta_\epsilon > 0$ such that*

$$\inf_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x(\xi_\delta < 1) > 1 - \epsilon.$$

The value of reducing to the unambiguous setting is that it allows us to study sequences of branching and killing events for deterministic configurations in order to make deductions about the possible behaviour of the BBB process. An *unambiguous configuration* consists of an unambiguous vector $x = (x_1, \dots, x_N)$ and a vector $w = (w_1, \dots, w_N)$ of non-negative integer weights with $\sum_{i=1}^N w_i = N$. A branching event for site $\ell \in [N]$ corresponds to increasing w_ℓ by 1, and a killing event for particle $k \in [N]$ corresponds to decreasing w_k by 1. Formally, given x, w as above, for $\ell \in [N]$ let

$$k = k(x, w, \ell) = \operatorname{argmax}_{j \in [N], w_j > 0} \left| x_j - \frac{1}{N+1} \left(\sum_{1 \leq i \leq N} w_i x_i + x_\ell \right) \right| \quad (1.6)$$

and define weights $g(x, w, \ell) = (g_1, \dots, g_N)$ by

$$g_j = w_j + \mathbb{1}_{\{j=\ell\}} - \mathbb{1}_{\{j=k\}}.$$

Note that $\sum_{i=1}^N g_i = N$, so the pair $x, g(x, w, \ell)$ again form an unambiguous configuration.

Given an unambiguous configuration x, w , any sequence l_1, \dots, l_m of elements of $[N]$ induces a sequence of weight vectors $w^{(0)}, w^{(1)}, \dots, w^{(m)}$, by letting particle l_i branch at step i . Formally, set $w^{(0)} = w$, and for $1 \leq i \leq m$ let $w^{(i)} = g(x, w^{(i-1)}, l_i)$. We say l_1, \dots, l_m is a valid sequence for x, w if $w_{l_i}^{(i-1)} > 0$ for all $1 \leq i \leq m$, i.e. if branching only occurs for particles with positive weight.

Lemma 1.6. *Given any unambiguous configuration x, w , there exists an integer $m \leq (N-1)^2$ and a sequence $l_1, \dots, l_m \in [N]$ which is valid for x, w such that $w^{(m)}$ contains exactly one non-zero entry.*

We say that a sequence l_1, \dots, l_m as in Lemma 1.6 *collapses* the (unambiguous) configuration x, w . The proof of Lemma 1.6, which is the key step in the proof of Harris recurrence, is found in Section 3.2, below.

To prove Harris recurrence for $\Phi = X - \bar{X}$, we now argue as follows. Step 1, we wait until the first time t that $X(t)$ is δ -unambiguous (for some small fixed $\delta > 0$); by Lemma 1.5 and the strong Markov property this takes a bounded amount of time. Step 2, we hope that in the time interval $[t, t + 1]$, (a) all particles stay within distance $\delta/4$ of their time- t locations, and (b) a sequence of branching events occurs which collapses the configuration $X(t), (1, 1, \dots, 1)$. (The all-ones vector represents the fact that at time t there is one particle at each location.) If Step 2 succeeds, then there is $j \in [N]$ such that in $X(t + 1)$ all particles lie within distance $\delta/4$ of $X_j(t)$, so all entries of $\Phi(t + 1)$ have distance at most δ from the origin.

By Lemma 1.6, there is at least one sequence consisting of at most $(N - 1)^2$ branching events which collapses the configuration, from which it is not hard to see that Step 2(b) succeeds with probability bounded away from zero uniformly over the possible values of $X(t)$. If Step 2 fails, we start over from Step 1. The Markov property then guarantees that both steps will succeed after a bounded number of trials. This means that from any initial configuration, Φ will reach a state where all particles are at bounded distance from the origin in bounded time. Combined with the fact that between branching events, the process X behaves like N independent Brownian motions, this allows us to prove Harris recurrence by comparing the law of the process Φ at a fixed time to the joint law of N independent d -dimensional standard Gaussians. The full details of this argument appear in Section 3.3.

1.3. Related work

The BBB model may be seen as a close relative of the Brunet-Derrida particle systems [9, 10] and their multidimensional generalizations [7]. A Brunet-Derrida particle system also consists of a population of a fixed size N undergoing both branching and spatial displacement, with individual trajectories taking values in \mathbb{R} . The law of the system is determined by a *fitness function* $s : \mathbb{R} \rightarrow \mathbb{R}$; when a branching event occurs, particles of lowest fitness are removed from the system to keep the population size constant. (The original papers on the subject [9, 10] considered discrete-time processes – branching random walks with selection, in which multiple branching events can occur simultaneously. Much of the subsequent work in the area has focused on branching Brownian motion, as in this paper.) The asymptotic displacements [3, 13], evolution of empirical particle densities [11, 12], and typical genealogies [4] of such systems have all received attention in the literature.

In [7], Berestycki and Zhao consider branching Brownian motion with selection in \mathbb{R}^d . The model is again specified by a fitness function, $s : \mathbb{R}^d \rightarrow \mathbb{R}$; a fixed number N of particles move as independent Brownian motions which independently branch at rate 1. On a branching event, a particle of minimal fitness is removed from the system. Berestycki and Zhao prove, among other results, the following facts. When $s(x) = |x|$, the large-time limit the particle system asymptotically travels at a constant speed, in a uniformly random direction, in a cloud of particles whose diameter is $O(1)$ in probability. When the fitness function is linear, $s(x) = \langle x, v \rangle$ for a fixed $v \in \mathbb{S}^{d-1}$, the cloud of particles asymptotically travels with constant speed in direction v . In this case (under some conditions on the time-zero configuration of the particles), the diameter of the cloud of particles in direction v is asymptotically $O(\log N)$ in probability, whereas for any $w \in \mathbb{S}^{d-1}$ with $\langle v, w \rangle = 0$, the diameter of the cloud in direction w is at least $O(\log N)^{3/2}$ in probability.

We also mention some recent and forthcoming works on branching systems with selection, with some similarity to ours.

- Berestycki, Brunet, Nolen and Penington [5, 6] study a branching Brownian motion $((X_i(t))_{i \in [N]}, t \geq 0)$ in \mathbb{R}^d with N particles, with a selection rule that can be specified via the fitness function $s(x) = -|x|$; in other words, on a branching event the particle farthest from the origin is removed. For such a model, [5, 6] proves results on the long-time and large-particle behaviour of the empirical distribution of particles in the system, by developing a connection with certain free-boundary problems. We discuss these works further in the conclusion, Section 4.

- The works [1, 2] consider binary branching Brownian motion $(X_i(t))_{i \in [\mathcal{N}(t)]}$ in \mathbb{R}^d with a variable number of particles, where the branching/death rate at position x at time t has the form

$$\mu_t(x) = g \left(\int_{\mathbb{R}} f(x-y) \nu_t(dy) \right);$$

here $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are Borel functions, and $\nu_t = \mathcal{N}(t)^{-1} \sum_{i=1}^{\mathcal{N}(t)} \delta_{X_i(t)}$ is the empirical measure of the time- t particle distribution. When $\mu_t(x)$ is positive it is interpreted as a branching rate; when this quantity is negative it is interpreted as a death rate. For certain choices of the functions f and g , Beckman [1] proves hydrodynamic limit theorems for such systems, showing that the evolution of the empirical density is governed by certain integro-differential equations; other models of the same sort are considered in [2].

1.4. Outline of the paper

The rest of the paper is organized as follows. Theorem 1.1 is a consequence of Proposition 1.4 for an appropriately defined sequence of times $(\tau_i; i \geq 1)$. In Section 2, we construct such a sequence $(\tau_i; i \geq 1)$ for the BBB process and state several key lemmas about these times, which we then use to demonstrate they satisfy the hypotheses of Proposition 1.4. We conclude Section 2 with the proof of Theorem 1.1. In Section 3, we prove the technical lemmas used in the construction of the times in Section 2, and also prove Harris Recurrence (Theorem 1.3), which follows relatively easily from one of these lemmas. Section 4 contains concluding remarks and open questions. Finally, in the Appendix (Section 5), we present the proof of Proposition 1.4.

2. Construction of $(\tau_i; i \geq 1)$ and the Proof of Theorem 1.1

Throughout this subsection, unless otherwise state, we assume that $N \geq 3$. The first portion of this section is devoted to the construction of $(\tau_i; i \geq 1)$, and to introducing two technical lemmas which motivate the construction. We thereafter show that $(\tau_i; i \geq 1)$ satisfies properties (1)-(5) of Proposition 1.4. We conclude with the proof of Theorem 1.1.

In fact, we will focus on constructing a sequence of regeneration times $(\rho_i; i \geq 1)$ for which the law of the BBB process has an explicit and fairly simple form (and, after recentering, is the same for all $i \geq 1$). We will then define $(\tau_i; i \geq 1)$ from these $(\rho_i; i \geq 1)$ (by letting $\tau_i := \rho_i - 1$), for which we will be able to verify properties (1)-(5) of Proposition 1.4.

2.1. Construction of $(\rho_i; i \geq 1)$ and some key ideas

For $I \subseteq \mathbb{N}$, given a collection $x = (x_i)_{i \in I}$ of vectors in \mathbb{R}^d , the *extent* of x is defined to be

$$E(x) := \sup_{i, j \in I} |x_j - x_i|.$$

For $t > 0$, we define an event A_t , measurable with respect to $\sigma(X(s), t \leq s \leq t+1)$, as follows. Write $G' := \{2, \dots, \lceil \frac{N+1}{2} \rceil\}$ and $D' := \{\lceil \frac{N+1}{2} \rceil + 1, \dots, N\}$, and note that $|G'| \geq |D'|$. Let $r_N := \frac{1}{4(N+1)}$. The event A_t occurs if and only if the following events all take place.

- (A_{t,1}) There are no branching events in the time interval $[t, t+1]$.
- (A_{t,2}) $X_j(t+1) - \bar{X}(t) \in B(-5e_1, r_N)$ for all $j \in G'$, and $X_j(t+1) - \bar{X}(t) \in B(5e_1, r_N)$ for all $j \in D'$.
- (A_{t,3}) $X_1(t+1) - \bar{X}(t) \in B(\gamma, r_N)$, where

$$\gamma := \frac{1}{N-1} \left(-5 \left\lceil \frac{N-1}{2} \right\rceil + 5 \left\lfloor \frac{N-1}{2} \right\rfloor \right) e_1 = \begin{cases} 0 & N \text{ odd,} \\ -\frac{5}{N-1} e_1 & N \text{ even.} \end{cases}$$

is roughly the position of the time- $(t+1)$ barycenter (reshifted by $\bar{X}(t)$) of the particles in $G' \cup D'$ along the direction of the standard unit vector $e_1 = (1, 0, \dots, 0)^T$.

In words, $A_t = A_{t,1} \cap A_{t,2} \cap A_{t,3}$ is the event that during the time interval $[t, t+1]$, no branchings occur and no single trajectory makes a large displacement; particle X_1 moves to a position close to the time- $(t+1)$ barycenter; and the remaining particles cluster into two roughly equal groups, both with small extent and both reasonably far from the barycenter and from each other. The shift by $\bar{X}(t)$ is simply an adjustment to make the event easier to describe (by ensuring that the starting particle configuration has barycenter located at 0). Moreover, the three clusters of particles with indices in G' , D' , and $\{1\}$ are approximately collinear.

Next, for $t \geq 0$, we define an event B_t , measurable with respect to $\sigma(X(s), t+1 \leq s \leq t+2)$, as follows. The event B_t occurs if and only if the following events both take place.

- ($B_{t,1}$) In the time interval $[t+1, t+2]$, particle X_1 and its descendants branch at least $N-1$ times and no other particle branches.
- ($B_{t,2}$) Every particle remains at distance less than or equal to r_N from the location of its time- $(t+1)$ ancestor from time $t+1$ to time $t+2$ or the time of its death, whichever comes first.

Although the event B_t only requires certain branching events to occur and minimal movement of the particles, we now show that the event $A_t \cap B_t$ forces all particles besides X_1 and its descendants to be killed in the time interval $[t+1, t+2]$.

We introduce \mathcal{S} , the set of all possible configurations $x \in \mathbb{R}^{d \times N}$ such that there exist disjoint sets G, C and D with $C \neq \emptyset$ and $G \cup C \cup D = [N]$ with

- (a) $x_i \in B(\gamma, 2r_N)$ for all $i \in C$, where γ was defined by ($A_{t,3}$).
- (b) $x_j \in B(-5e_1, 2r_N)$ for all $j \in G$, and $x_j \in B(5e_1, 2r_N)$ for all $j \in D$.
- (c) $|G \cup C| \geq \lceil \frac{N-1}{2} \rceil + 1$ and $|D \cup C| \geq \lfloor \frac{N-1}{2} \rfloor + 1$.

Conditioned on the event $A_t \cap B_t$, we note that $X(s) - \bar{X}(t) \in \mathcal{S}$ for all $s \in [t+1, t+2]$. Therefore, to determine which particles of X get killed in time $[t+1, t+2]$, it suffices to show that for all configurations belonging to \mathcal{S} , a branching event in C implies that, under the same killing rules as BBB (namely that the particle farthest away from the barycenter is killed), the killing event occurs in $G \sqcup D$.

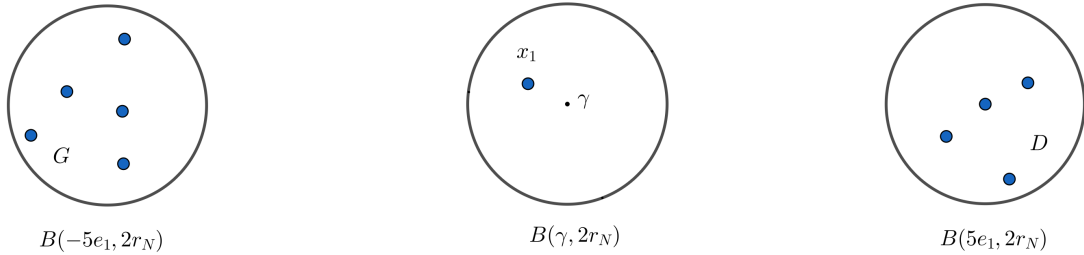


Figure 1. Example of particle configuration $x = (x_i)_{i \in [N]}$ in \mathcal{S} with $C = \{1\}$ which could arise just after event A has occurred. The particles in the ball on the left are $(x_i)_{i \in G}$, and those in the ball on the right are $(x_i)_{i \in D}$.

Lemma 2.1. *For any configuration $x \in \mathcal{S}$, a branching event in C implies that the killed particle belongs to $G \sqcup D$.*

Proof. Suppose first that $G, D \neq \emptyset$ and that the particle with index $\ell \in C$ branches. Let \bar{x}^{N+1} denote the barycenter of the system with $N+1$ particles (before a killing event occurs). Then \bar{x}^{N+1} lies in the convex hull of the particles $\{x_j\}_{j \in [N]} \cup \{x_\ell\}$. Since the clusters of particles $G, C,$

and D are roughly collinear with C located in between G and D , it follows that the resulting killing occurs in either G or D .

We may therefore assume without loss of generality that $D = \emptyset$, $|C| \geq \lfloor \frac{N-1}{2} \rfloor + 1$, and $|G| \leq \lfloor \frac{N-1}{2} \rfloor$. Upon a branching event in C , we observe that for the system with $N+1$ particles (before a killing occurs), $|C| > |G|$. A small computation then shows that $\bar{x}^{(N+1)}$ lies in the halfspace $\{y \in \mathbb{R}^d : y \cdot e_1 \geq \iota - 2r_N\}$, where

$$\iota := \frac{1}{N+1} \left[-5 \left\lfloor \frac{N-1}{2} \right\rfloor - \left(\left\lfloor \frac{N-1}{2} \right\rfloor + 2 \right) \gamma \right] = \begin{cases} -\frac{5}{2} \frac{N-1}{N+1} & N \geq 3 \text{ odd,} \\ -\frac{5}{2} \frac{N^2+2}{N^2-1} & N \geq 3 \text{ even.} \end{cases}$$

The point ιe_1 roughly corresponds to the barycenter of the $(N+1)$ system of particles after a branching event in C . Using now that for all $i \in G$, $x_i \in B(-5e_1, 2r_N)$ and for all $j \in C$, $x_j \in B(\gamma, 2r_N)$, still with $r_N = \frac{1}{4(N+1)}$, we see that

$$\min_{i \in G'} |x_i - \bar{x}^{(N+1)}| > \max_{j \in C'} |x_j - \bar{x}^{(N+1)}|.$$

Therefore the particle being killed must belong to G . □

Remark 2.2. *Conditioned on $A_t \cap B_t$, since $X(s) - \bar{X}(t) \in \mathcal{S}$ for all $s \in [t+1, t+2]$, it follows by Lemma 2.1 that all particles with indices in $G \sqcup D$ to die, so at time $t+2$ all particles are descendants of $X_1(t+1)$. The resulting particle configuration looks like the one shown in Figure 2.*

Remark 2.3. *The proof of Lemma 2.1 can be seen as a simple case of using “deterministic configurations” to deduce which killing events occur, when the particles are allowed very limited movement. A more complex formulation of similar ideas appears in the proof of Lemma 2.5.*

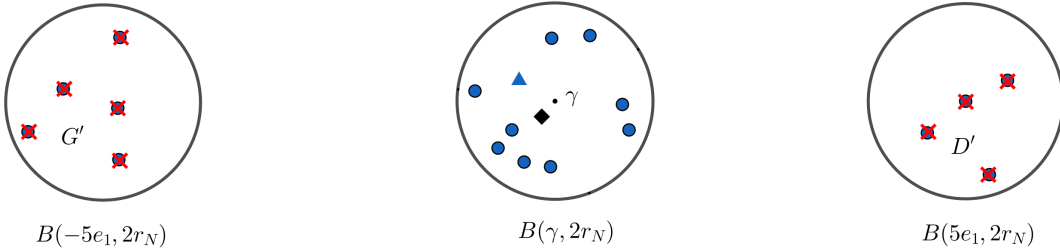


Figure 2. Example of a time- $(t+2)$ configuration after $A_t \cap B_t$ has occurred. The crossed-out particles indicate that when $A_t \cap B_t$ occurs, at time $t+2$, the particles in $(X_i(t+1))_{i \in G' \cup D'}$ live in $B(\gamma, 2r_N)$ (by Lemma 2.1). The blue triangle is the location of $X_1(t+2) - \bar{X}(t)$, and the black square is the location of $\bar{X}(t+2) - \bar{X}(t)$.

Observe that the events A_t and B_t are invariant under translations in \mathbb{R}^d . Namely, A_t and B_t occur for $X = (X_i)_{i \in [N]}$ if and only if they occur for $X + y := (X_i + y)_{i \in [N]}$ for every $y \in \mathbb{R}^d$.

For any $L > 0$, define

$$C_L := \{x \in \mathbb{R}^{d \times N} : E(x) \leq L\},$$

the set of deterministic N -particle configurations $(x_i)_{i \in [N]}$ with extent at most L . Denote $A := A_0$ and $B := B_0$. It turns out that as long as we start from configurations of bounded extent, there is a uniformly positive probability that $A \cap B$ occurs.

Lemma 2.4. *For all $L > 0$, there exists $\epsilon = \epsilon_{L,N} > 0$ such that*

$$\inf_{x \in C_L} \mathbf{P}_x(A \cap B) > \epsilon.$$

Since the lower bound in Lemma 2.4 depends on L , we will want to show that regardless of the extent of the starting configuration, the BBB reaches a state with extent L in finite time. This is a consequence of the next lemma.

Lemma 2.5. For $T^{(L)} := \inf\{t \geq 0 : E(X(t)) \leq L\}$, we have

$$\lim_{L \rightarrow \infty} \sup_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x \left(T^{(L)} > L \right) = 0.$$

Remark 2.6. (i) Lemma 2.5 holds for all $N \geq 1$. The cases $N = 1, 2$ are trivial, and the case $N \geq 3$ will be proved in Section 3.3.

(ii) It is a direct consequence of Lemma 2.5 that $T^{(L)}$, the time to reach a configuration with extent at most L , has exponential tails. Indeed, it suffices to choose $L > 0$ such that $\sup_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x(T^{(L)} > L) < \frac{1}{2}$ and then apply the Markov property.

We will see that Lemma 2.5 implies Harris recurrence (Theorem 1.3). Equipped with Lemmas 2.4 and 2.5, we can now construct the stopping times $(\rho_i; i \geq 1)$ as follows. Fix L as in Remark 2.6 (ii) and set $T_1 := \inf\{t \geq 1 : E(X(t)) \leq L\}$. For each $i \geq 1$, recursively define the stopping times

$$T_{i+1} := \inf\{t \geq T_i + 2 : E(X(t)) \leq L\}, \quad i \geq 1, \quad (2.1)$$

and let

$$I_1 := \inf\{i \geq 1 : A_{T_i} \cap B_{T_i} \text{ occurs}\}.$$

By Remark 2.6 (ii), and the strong Markov property, $T_i < \infty$ almost surely for each $i \geq 1$. By the strong Markov property and Lemma 2.4, $A_{T_i} \cap B_{T_i}$ has positive probability uniformly over all possible initial configurations for each $i \geq 1$. In particular, an application of the second Borel-Cantelli lemma implies that $I_1 < \infty$ \mathbf{P}_x -almost surely. Thus, we can define $\tau_1 := T_{I_1} + 1$ and $\rho_1 := T_{I_1} + 2$; both τ_1 and ρ_1 are \mathbf{P}_x -a.s. finite. By construction, we observe that at time ρ_1 , the events $A_{T_{I_1}}$ and $B_{T_{I_1}}$ have occurred (as depicted in Figure 2), and at time $\rho_1 - 2$, the particles had extent at most L .

Next, for each integer $l > 1$, inductively set

$$I_l := \inf\{i > I_{l-1} : A_{T_i} \cap B_{T_i} \text{ occurs}\},$$

and let

$$\begin{aligned} \tau_l &:= T_{I_{l-1}} + 1, \text{ and} \\ \rho_l &:= T_{I_{l-1}} + 2. \end{aligned} \quad (2.2)$$

At time τ_l , the event $A_{T_{I_{l-1}}}$ has just occurred, and the event $B_{T_{I_{l-1}}}$ is about to occur; by time ρ_l the event $B_{T_{I_{l-1}}}$ has also occurred. Note that both $(\tau_i; i \geq 1)$ and $(\rho_i; i \geq 1)$ are strictly increasing sequences, and that $(\rho_i; i \geq 1)$ are stopping times with respect to the natural filtration of X . We now have the following key lemma, which guarantees that the times $(\tau_i; i \geq 1)$ satisfy the hypotheses of Proposition 1.4.

Lemma 2.7. Assume $N \geq 3$. Let $(\tau_i; i \geq 1)$ be defined as above. Then for any $x \in \mathbb{R}^{d \times N}$, under \mathbf{P}_x , X and $(\tau_i; i \geq 1)$ satisfy the hypotheses of Proposition 1.4.

To prove Lemma 2.7, we must verify properties (1)-(5) of Proposition 1.4. Lemma 2.7 follows immediately the results of the next two subsections. Specifically, Corollary 2.9, Proposition 2.11 and Proposition 2.13 establish properties (1)-(3), property (4), and property (5) from the hypotheses of Proposition 1.4, respectively.

2.2. The regeneration structure

The goal of this section is to show that, when considered on time intervals of the form $[\tau_m, \tau_{m+1}]$, the BBB process has an essentially IID structure. The key to revealing this structure is to only focus on descendants of particles of $X_1(\tau_m)$. Ignoring other particles creates some indexing issues, which have to be managed in order to avoid creating subtle dependencies. However, the gist of the story

is that the events A_{τ_m-1} and B_{τ_m-1} together ensure that the descendants of particle 1 take over the population on the time interval $[\tau_m, \tau_m + 1]$; this is what yields the IID structure.

Fix any $m \geq 1$, write $\sigma_m(1) = 1$, and for $2 \leq i \leq N$ let $\sigma_m(i)$ be the index of the $(i-1)$ 'st particle to be killed in the time interval $[\tau_m, \tau_m + 1]$. Because the event $A_{\tau_m-1} \cap B_{\tau_m-1}$ occurs, all particles aside from X_1 die during this time interval (Remark 2.2), so $(\sigma_m(i), 1 \leq i \leq N)$ is a permutation of $[N]$.

Next, for $s \geq 0$, let $N_m(s)$ be the number of living time- $(\tau_m + s)$ descendants of $X_1(\tau_m)$, including $X_1(\tau_m + s)$ itself. The event $B_{\tau_m-1} = B_{T_{I_m}}$ ensures that $N_m(s) \geq 1$ for $s \geq 0$ and that $N_m(1) = N$, so the descendants of $X_1(\tau_m)$ have taken over the population by time $\tau_m + 1 = \rho_m$. It follows that $N_m(s) = N$ for all $s \geq 1$.

Now write $\hat{X}^{(m)}(s) = (\hat{X}_j^{(m)}(s))_{j \in [N_m(s)]}$ for the locations of time- $(\tau_m + s)$ descendants of $X_1(\tau_m)$, listed in order of their birth times; formally, $\hat{X}_j^{(m)}(s) = X_{\sigma_m(j)}(\tau_m + s)$. This means that for all $s \geq 0$,

$$(\hat{X}_j^{(m)}(s) - \hat{X}_1^{(m)}(0))_{j \in [N_m(s)]} = (X_{\sigma_m(j)}(\tau_m + s) - X_1(\tau_m))_{j \in [N_m(s)]}.$$

Write $\mathbf{X}^{(m)} = ((\hat{X}_j^{(m)}(s) - \hat{X}_1^{(m)}(0))_{j \in [N_m(s)]}, 0 \leq s \leq \tau_{m+1} - \tau_m)$. The process $\mathbf{X}^{(m)}$ describes the locations of descendants of $X_1(\tau_m)$ relative to that of $X_1(\tau_m)$, between time τ_m and τ_{m+1} .

Finally, recall from Section 1.1 that we allow for the BBB process to be started from a configuration with fewer than N particles, and in this case we write $N(t)$ for the number of particles alive at time t . It is useful for the next proposition to introduce the notation $\tilde{X} := (\tilde{X}(t), t \geq 0)$ with $\tilde{X}(t) = (\tilde{X}_i(t))_{i \in [N(t)]}$, for a BBB process started from a single particle at $0 \in \mathbb{R}^d$. We then write F for the event that $\sup_{0 \leq s \leq 1} \max_{1 \leq i \leq N(t)} |\tilde{X}_i(s)| \leq r_N$ and that $N(1) = N$.

Proposition 2.8. *The processes $(\mathbf{X}^{(m)}; m \geq 1)$ are iid, and their common distribution is the conditional distribution of $(\tilde{X}(t), 0 \leq t \leq \tau_1)$ given F .*

Proof. The main subtlety to the proof is that the events $A_{\tau_m-1} \cap B_{\tau_m-1}$ contain some information about the process on the time interval $[\tau_m, \rho_m] = [\tau_m, \tau_m + 1]$, and the effect of this on the law must be understood. Introduce the filtration $(\mathcal{H}_t, t \geq 0)$ defined by

$$\mathcal{H}_t := \sigma \left((X(s), 0 \leq s \leq t), (\mathbb{1}_{A_{T_i} \cap B_{T_i}}, 0 \leq T_i \leq t-1) \right).$$

For any $m \geq 1$, by definition, $\tau_m \leq t$ if and only if $\rho_m \leq t+1$, which in turn occurs if and only if $T_{I_m} \leq t-1$ and $A_{T_{I_m}} \cap B_{T_{I_m}}$ occurs. It follows that τ_m is an \mathcal{H}_t -stopping time for all $m \geq 1$.

The event A_{τ_m-1} is $\sigma(X(s), 0 \leq s \leq \tau_m)$ -measurable. The event $B_{\tau_m-1,1} \supset B_{\tau_m-1}$ enforces that $X_1(\tau_m)$ and its descendants branch at least $N-1$ times in the time interval $[\tau_m, \tau_{m+1}]$. Lemma 2.1 and Remark 2.2 together show that when A_{τ_m-1} and B_{τ_m-1} both occur, each such branching event increases the number of living descendants of $X_1(\tau_m)$ until those descendants occupy the full population. In particular, at time $\tau_m + 1 = T_{I_m} + 2$, all individuals are descendants of $X_1(\tau_m)$, and so $N_m(1) = N$.

Moreover, the only constraint on the motion of particles during the time interval $[\tau_m, \tau_{m+1}]$ is imposed by the event $B_{\tau_m-1,2} \supset B_{\tau_m-1}$; this event precisely requires that $|X_j^{(m)}(s)| = |\hat{X}_j^{(m)}(s) - \hat{X}_1^{(m)}(0)| \leq r_N$ for all $0 \leq s \leq 1$ and all $j \in [N_m(s)]$.

The preceding paragraphs show that $(X^{(m)}(s), 0 \leq s \leq 1)$ is distributed as a BBB process run for time 1, started from a single particle at $0 \in \mathbb{R}^d$, and conditioned on the event F defined in the proposition statement. To understand the law of the whole process $\mathbf{X}^{(m)} = (X^{(m)}(s), 0 \leq s \leq \tau_{m+1} - \tau_m)$, recall that the event $A_{\tau_m-1} \cap B_{\tau_m-1}$ is measurable with respect to $\mathcal{F}_{\tau_{m+1}} = \mathcal{F}_{T_{I_m}+2}$. It follows that the process

$$((\hat{X}_j^{(m)}(s) - \hat{X}_1^{(m)}(0))_{j \in [N]}, s \geq 0)$$

is distributed as a BBB started from a single particle at $0 \in \mathbb{R}^d$ and conditioned on the event F .

The last thing to check is that $\tau_{m+1} - \tau_m$ indeed has the correct law. Since $\tau_m = T_{I_m} + 1$, necessarily $\tau_{m+1} \geq T_{I_{m+1}} \geq T_{I_m} + 2 = \tau_m + 1$. Our approach is thus to first understand the law of $T_{I_{m+1}}$, so that we can think of the time interval $[\tau_m, \tau_{m+1}]$ as $[\tau_m, T_{I_{m+1}}] \cup [T_{I_{m+1}}, \tau_{m+1}]$. By definition, $T_{I_{m+1}} \geq T_{I_m} + 2 = \tau_m + 1$; thus

$$T_{I_{m+1}} - \tau_m = \inf\{t \geq 1 : E(\hat{X}^{(m)}(t)) \leq L\}.$$

Since

$$T_1 = \inf\{t \geq 1 : E(X(t)) \leq L\},$$

it follows that the law of $T_{I_{m+1}} - \tau_m$ is precisely the conditional law of T_1 in a BBB started from a single particle at $0 \in \mathbb{R}^d$ given that the event F occurs. The fact that $\tau_{m+1} - \tau_m$ and $((\hat{X}_j^{(m)}(s) - \hat{X}_1^{(m)}(0))_{j \in [N]}, s \geq 0)$ have the correct joint distribution now follows by the strong Markov property applied at time $T_{I_{m+1}}$ together with the definition of τ_{m+1} . \square

Corollary 2.9. *For any $x \in \mathbb{R}^{d \times N}$, under \mathbf{P}_x , the BBB process and the stopping times $(\tau_i, i \geq 1)$ satisfy conditions (1), (2) and (3) of Proposition 1.4.*

Proof. The differences $(\tau_{m+1} - \tau_m, m \geq 1)$ are the total lifetimes of the processes $(X^{(m)}; m \geq 1)$; since these processes are IID, so are the differences $(\tau_{m+1} - \tau_m, m \geq 1)$. This verifies property (1).

Next, writing $X_1^{(m)} = (X_1^{(m)}(s), 0 \leq s \leq \tau_{m+1} - \tau_m)$ for the projection of $X^{(m)}$ on its first coordinate, then the processes $(X_1^{(m)}, m \geq 1)$ are also IID. By definition,

$$X_1^{(m)}(s) = X_{\sigma_m(1)}(\tau_m + s) - X_1(\tau_m) = X_1(\tau_m + s) - X_1(\tau_m),$$

the second inequality since $\sigma_m(1) = 1$ for all $m \geq 1$. It follows that the processes

$$(X_1(\tau_m + s) - X_1(\tau_m), 0 \leq s \leq \tau_{m+1} - \tau_m)$$

are IID for $m \geq 1$; properties (2) and (3) of the proposition are immediate. \square

2.3. Finite variance of the increments

In this section, we prove property (4) in Proposition 1.4. We first establish that the regeneration increments have exponential tails. Recall that $\rho_1 := \tau_1 + 1$.

Lemma 2.10. *There exists $b = b_{L,N} > 1$ such that*

$$\sup_{x \in \mathbb{R}^{d \times N}} \mathbf{E}_x[b^{\rho_1}] < \infty.$$

Proof. Fix any initial configuration $x \in \mathbb{R}^{d \times N}$. For any $b > 1$, using the change of variables $r := \log_b(s)$, we may write

$$\begin{aligned} \mathbf{E}_x[b^{\rho_1}] &= \int_0^\infty \mathbf{P}_x(\rho_1 > \log_b(s)) ds \\ &= \ln(b) \left(\int_{-\infty}^1 b^r \mathbf{P}_x(\rho_1 > r) dr + \int_1^\infty b^r \mathbf{P}_x(\rho_1 > r) dr \right). \end{aligned}$$

The first integral is bounded from above by $\ln(b) \int_{-\infty}^1 b^r dr$ which is finite for any $b > 1$. By Lemma 2.5, there exists $L > 1$ such that

$$\inf_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x(T^{(L)} \leq L) > 1/2. \quad (2.3)$$

Moreover, we claim Lemma 2.4 implies the existence of a constant $c = c_{L,N} > 0$ such that for all $t \geq L + 2$,

$$\inf_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x(\rho_1 \leq t | T^{(L)} \leq L) \geq c. \quad (2.4)$$

Indeed, since $T^{(L)} = T_1$, we obtain that for all $t \geq L + 2$, for any $x \in \mathbb{R}^{d \times N}$,

$$\begin{aligned}
\mathbf{P}_x(\rho_1 \leq t | T_1 \leq L) &= \mathbf{E}_x[\mathbb{1}_{\{\rho_1 \leq t\}} | T_1 \leq L] \\
&= \mathbf{P}_x(T_1 \leq L)^{-1} \mathbf{E}_x[\mathbb{1}_{\{\rho_1 \leq t\}} \mathbb{1}_{\{T_1 \leq L\}}] \\
&= \mathbf{P}_x(T_1 \leq L)^{-1} \mathbf{E}_x[\mathbf{E}_x[\mathbb{1}_{\{\rho_1 \leq t\}} | \mathcal{F}_{T_1}] \mathbb{1}_{\{T_1 \leq L\}}] \\
&= \mathbf{P}_x(T_1 \leq L)^{-1} \mathbf{E}_x[\mathbf{P}_{X(T_1)}(\rho_1 \leq t - T_1) \mathbb{1}_{\{T_1 \leq L\}}] \text{ (strong Markov prop.)} \\
&= \mathbf{E}_x[\mathbf{P}_{X(T_1)}(\rho_1 \leq t - T_1) | T_1 \leq L] \\
&\geq \inf_{y \in C_L} \mathbf{P}_y(A \cap B) \\
&= c > 0.
\end{aligned}$$

Denote $c' := c/2 \in (0, 1)$. By (2.3) and (2.4), for any $t \geq L + 2$ and any $x \in \mathbb{R}^{d \times N}$,

$$\begin{aligned}
\mathbf{P}_x(\rho_1 \leq t) &\geq \mathbf{P}_x(\rho_1 \leq t, T^{(L)} \leq L) \\
&\geq \mathbf{P}_x(\rho_1 \leq t | T^{(L)} \leq L) \mathbf{P}_x(T^{(L)} \leq L) \\
&> c',
\end{aligned}$$

whence for each $x \in \mathbb{R}^{d \times N}$,

$$\mathbf{P}_x(\rho_1 > t) < 1 - c'.$$

It follows by the strong Markov property and time-homogeneity that

$$\begin{aligned}
\mathbf{P}_x(\rho_1 > t) &\leq \prod_{k=1}^{\lfloor t/(L+2) \rfloor} \mathbf{P}_x(\rho_1 > k(L+2) | \rho_1 > (k-1)(L+2)) \\
&\leq \left(\sup_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x(\rho_1 > L+2) \right)^{\lfloor t/(L+2) \rfloor - 1} \\
&\leq (1 - c')^{\lfloor t/(L+2) \rfloor - 1}
\end{aligned}$$

for all $x \in \mathbb{R}^{d \times N}$. Thus,

$$\int_1^\infty b^r \mathbf{P}_x(\rho_1 > r) dr \leq \int_1^\infty b^r (1 - c')^{\lfloor r/(L+2) \rfloor - 1} dr.$$

The integral is finite provided we choose $b > 1$ such that $b(1 - c')^{1/(L+2)} < 1$. Since $c' \in (0, 1)$, this is possible. \square

Note that, taking $\rho_0 \equiv 0$, the lemma implies that

$$\sup_{x \in \mathbb{R}^{d \times N}} \mathbf{E}_x[b^{\rho_{i+1} - \rho_i}] < \infty$$

for any $i \geq 0$, since we may write by the tower law and the strong Markov property

$$\mathbf{E}_x[b^{\rho_{i+1} - \rho_i}] = \mathbf{E}_x[\mathbf{E}_x[b^{\rho_{i+1} - \rho_i} | \mathcal{F}_{\rho_i}]] = \mathbf{E}_x[\mathbf{E}_{X(\rho_i)}[b^{\rho_1 - \rho_0}]] = \mathbf{E}_x[\mathbf{E}_{X(\rho_i)}[b^{\rho_1}]]$$

and Lemma 2.10 implies that the inner conditional expectation is deterministically bounded by some $C < \infty$. Also observe that $\tau_{i+1} - \tau_i = \rho_{i+1} - \rho_i$ for each $i \geq 1$ and $\tau_1 \leq \rho_1$, so Lemma 2.10 yields

$$\sup_{x \in \mathbb{R}^{d \times N}} \mathbf{E}_x[b^{\tau_{i+1} - \tau_i}] < \infty. \quad (2.5)$$

for all $i \geq 0$, where we have defined $\tau_0 \equiv 0$. This last fact will be used in the proof of property (4) of Proposition 1.4 for the BBB process (Proposition 2.11).

Proposition 2.11. *For any $x \in \mathbb{R}^{d \times N}$, under \mathbf{P}_x , the BBB process and the stopping times $(\tau_i, i \geq 1)$ satisfy condition (4) of Proposition 1.4.*

Proof. The condition asserts that

$$\mathbf{E}_x \left[\sup_{\tau_1 \leq t \leq \tau_2} |X_1(t) - X_1(\tau_1)|^2 \right] < \infty.$$

We bound the supremum as follows. Recall that $\rho_1 = \tau_1 + 1$, and write

$$\begin{aligned} \sup_{\tau_1 \leq t \leq \tau_2} |X_1(t) - X_1(\tau_1)|^2 &= \max \left(\sup_{\tau_1 \leq t \leq \rho_1} |X_1(t) - X_1(\tau_1)|^2, \sup_{\rho_1 \leq t \leq \tau_2} |X_1(t) - X_1(\tau_1)|^2 \right) \\ &\leq \sup_{\tau_1 \leq t \leq \rho_1} |X_1(t) - X_1(\tau_1)|^2 + \sup_{\rho_1 \leq t \leq \tau_2} |X_1(t) - X_1(\tau_1)|^2. \\ &\leq r_N + \sup_{\rho_1 \leq t \leq \tau_2} |X_1(t) - X_1(\tau_1)|^2, \end{aligned}$$

the last inequality since by definition, $|X_1(t) - X_1(\tau_1)| \leq r_N$ for $t \leq \tau_1 + 1 = \rho_1$. So to prove the lemma it suffices to show that

$$\mathbf{E}_x \left[\sup_{\rho_1 \leq t \leq \tau_2} |X_1(t) - X_1(\tau_1)|^2 \right] < \infty.$$

We further decompose this supremum by writing

$$\sup_{\rho_1 \leq t \leq \tau_2} |X_1(t) - X_1(\tau_1)|^2 \leq 4 \left(r_N + \sup_{\rho_1 \leq t \leq \tau_2} |X_1(t) - X_1(\tau_1)|^2 \right),$$

where we have used the inequality $(a - b)^2 \leq 4(a - c)^2 + 4(c - b)^2$ and the fact that $|X_1(\rho_1) - X_1(\tau_1)| \leq r_N$. It thus suffices to prove that

$$\mathbf{E}_x \left[\sup_{\rho_1 \leq t \leq \tau_2} |X_1(t) - X_1(\tau_1)|^2 \right] < \infty.$$

Now write π for the law of $X(\tau_1 + 1) - X_1(\tau_1)$. Precisely, if \mathbf{X} is the BBB process started from a single particle at $0 \in \mathbb{R}^d$, then π is the conditional law of $X_1(1)$ given that $N(1) = N$ and that

$$\sup_{0 \leq t \leq 1} \max_{1 \leq i \leq N(t)} |X_i(t)| \leq r_N.$$

Recall that $\tau_1 - 1 = T_{I_1}$ is the time at which $A_{T_{I_1}} \cap B_{T_{I_1}}$ occurs. By definition, $A_{T_{I_1}} \cap B_{T_{I_1}}$ is \mathcal{F}_{ρ_1} -measurable, since $\rho_1 = \tau_1 + 1 = T_{I_1} + 2$. It follows by the strong Markov property that

$$\mathbf{E}_x \left[\sup_{\rho_1 \leq t \leq \tau_2} |X_1(t) - X_1(\tau_1)|^2 \right] = \mathbf{E}_\pi \left[\sup_{0 \leq t \leq \tau_1} |X_1(t) - X_1(0)|^2 \right],$$

and we must show the final expectation is finite. To do so, make use of the first construction of the BBB process given in Section 1.1. In other words, we view \mathbf{X} as constructed from N independent Brownian motions $(B_i, i \in [N])$ with the aid of a Poisson counting process \mathcal{J} which gives the branching times, and of a sequence $(\mathcal{U}_i; i \geq 1)$ of random variables which determines which particle branches at each branching time. An immediate consequence of this construction is that

$$\sup_{0 \leq t \leq \tau_1} \max_{j \in [N]} |X_j(t) - X_j(0)| \leq \mathcal{J}_{\tau_1} \sup_{0 \leq s \leq \tau_1} \max_{j \in [N]} |B_j(s)|, \quad (2.6)$$

so it suffices to prove that

$$\mathbf{E}_\pi \left[\mathcal{J}_{\tau_1}^2 \sup_{0 \leq s \leq \tau_1} \max_{j \in [N]} |B_j(s)|^2 \right] < \infty;$$

establishing this bound occupies the remainder of the proof.

By the Cauchy-Schwarz inequality, we have

$$\mathbf{E}_\pi \left[\mathcal{J}_{\tau_1}^2 \sup_{0 \leq s \leq \tau_1} \max_{j \in [N]} |B_j(s)|^2 \right] \leq \mathbf{E}_\pi [\mathcal{J}_{\tau_1}^4]^{1/2} \mathbf{E}_\pi \left[\sup_{0 \leq s \leq \tau_1} \max_{j \in [N]} |B_j(s)|^4 \right]^{1/2} \quad (2.7)$$

Moreover, since $t \mapsto \mathcal{J}_t$ is almost-surely nondecreasing

$$\begin{aligned} \mathbf{E}_\pi [\mathcal{J}_{\tau_1}^4] &= \mathbf{E}_\pi \left[\sum_{p=0}^{\infty} \mathcal{J}_{\tau_1}^4 \mathbb{1}_{\{\tau_1 \in [2^p, 2^{p+1}]\}} \right] \\ &\leq \sum_{p=0}^{\infty} \mathbf{E}_\pi [\mathcal{J}_{2^{p+1}}^4 \mathbb{1}_{\{\tau_1 \in [2^p, 2^{p+1}]\}}] \\ &\leq \sum_{p=0}^{\infty} \mathbf{E}_\pi [\mathcal{J}_{2^{p+1}}^8]^{1/2} \mathbf{P}_\pi (\tau_1 \geq 2^p)^{1/2}. \end{aligned} \quad (2.8)$$

By Markov's inequality and (2.5), there exist constants $b > 1$ and $K > 0$ such that

$$\mathbf{P}_\pi (\tau_1 \geq 2^p) \leq Kb^{-2^p}. \quad (2.9)$$

A standard computation of higher moments of Poisson random variables yields

$$\mathbf{E}_\pi [\mathcal{J}_{2^{p+1}}^8] = \sum_{i=0}^8 N^i 2^{i(p+1)} \left\{ \begin{matrix} 8 \\ i \end{matrix} \right\} = O(N^8 2^{8(p+1)}),$$

where $\left\{ \begin{matrix} j \\ i \end{matrix} \right\}$ denotes a Stirling number of the second kind. Continuing from (2.8), we obtain

$$\mathbf{E}_\pi [\mathcal{J}_{\tau_1}^4] \leq K^{1/2} N^4 \sum_{p=0}^{\infty} O(2^{4(p+1)}) b^{-2^{p-1}} < \infty. \quad (2.10)$$

To bound the second factor in (2.7), we note that

$$\begin{aligned} \mathbf{E}_\pi \left[\sup_{0 \leq s \leq \tau_1} \max_{i \in [N]} |B_i(s)|^4 \right] &= \sum_{p=0}^{\infty} \mathbf{E}_\pi \left[\sup_{0 \leq s \leq \tau_1} \max_{i \in [N]} |B_i(s)|^4 \mathbb{1}_{\{\tau_1 \in [2^p, 2^{p+1}]\}} \right] \\ &\leq \sum_{p=0}^{\infty} \mathbf{E}_\pi \left[\sup_{0 \leq s \leq 2^{p+1}} \max_{i \in [N]} |B_i(s)|^4 \mathbb{1}_{\{\tau_1 \in [2^p, 2^{p+1}]\}} \right] \\ &\leq \sum_{p=0}^{\infty} \mathbf{P}_\pi (\tau_1 \geq 2^p)^{1/2} \mathbf{E}_\pi \left[\sup_{0 \leq s \leq 2^{p+1}} \max_{i \in [N]} |B_i(s)|^8 \right]^{1/2}. \end{aligned} \quad (2.11)$$

Observe that

$$\max_{i \in [N]} |B_i(s)|^8 \leq d^4 \max_{i \in [N]} \max_{j \in [d]} |B_{ij}(s)|^8,$$

and that

$$\left(\max_{i \in [N]} \max_{j \in [d]} |B_{ij}(t)|, t \geq 0 \right)$$

is a submartingale. So Doob's L^8 inequality and IID-ness of the one-dimensional Brownian motions B_{ij} imply that

$$\begin{aligned} \mathbf{E}_\pi \left[\sup_{0 \leq s \leq 2^{p+1}} \max_{i \in [N]} |B_i(s)|^8 \right] &\leq d^4 \left(\frac{8}{7} \right)^8 \mathbf{E}_\pi \left[\max_{i \in [N]} \max_{j \in [d]} |B_{ij}(2^{p+1})|^8 \right] \\ &\leq d^5 N \left(\frac{8}{7} \right)^8 \mathbf{E}_\pi [|B_{11}(2^{p+1})|^8] \\ &= d^5 N O(2^{4(p+1)}), \end{aligned} \quad (2.12)$$

where we have used that $\mathbf{E}_\pi [|B_{11}(2^{p+1})|^8] = O(2^{4(p+1)})$. Combining (2.11), (2.12) and (2.9), we conclude that

$$\mathbf{E}_\pi \left[\sup_{0 \leq s \leq \tau_1} \max_{i \in [N]} |B_i(s)|^4 \right] \leq K^{1/2} d^5 N \sum_{p=0}^{\infty} b^{-2p-1} O(2^{4(p+1)}) < \infty.$$

as required. \square

2.4. Barycenter approximation in probability

In this section, we show that property (5) of Proposition 1.4 holds for the BBB process. We will make frequent use of the following fact on the convergence in probability of certain random variables.

Fact 2.12. *Let $(k_m)_{m \geq 1}$ be a sequence of nonnegative integer valued random variables with the property that there exists a constant $\kappa > 0$ such that $m^{-1}k_m \rightarrow \kappa$ in probability as $m \rightarrow \infty$. Suppose that $(Z_i)_{i \geq 1}$ are identically distributed nonnegative random variables with $\mathbf{E}[Z_1^2] < \infty$. Then, as $m \rightarrow \infty$*

$$m^{-1/2} \max_{1 \leq i < k_m} Z_i \rightarrow 0 \quad (2.13)$$

in probability.

Proof. Since $Z_1 \in L^2(\mathbf{P})$, it is \mathbf{P} -almost surely finite. Hence, by Chebyshev's inequality and the dominated convergence theorem, we obtain

$$\lim_{m \rightarrow \infty} m \mathbf{P}(m^{-1/2} Z_1 > \epsilon) \leq \lim_{m \rightarrow \infty} \mathbf{E} [\epsilon^{-2} Z_1^2 \mathbb{1}_{m^{-1/2} Z_1 > \epsilon}] = 0 \quad (2.14)$$

We have

$$\begin{aligned} & \mathbf{P} \left(m^{-1/2} \max_{1 \leq i < k_m} Z_i > \epsilon \right) \\ & \leq \mathbf{P} \left(m^{-1/2} \max_{1 \leq i < \epsilon^{-1} m} Z_i > \epsilon \mid k_m \leq \epsilon^{-1} m \right) + \mathbf{P} (k_m > \epsilon^{-1} m). \end{aligned} \quad (2.15)$$

For all $\epsilon > 0$ small enough that $\epsilon^{-1} > \kappa$, since $k_m \stackrel{m \rightarrow \infty}{\sim} \kappa m$ \mathbf{P} -a.s. by assumption, it follows that

$$\lim_{m \rightarrow \infty} \mathbf{P} (k_m > \epsilon^{-1} m) = 0. \quad (2.16)$$

Moreover, since the Z_i are identically distributed, a union bound and (2.14) give

$$\mathbf{P} \left(m^{-1/2} \max_{1 \leq i < \epsilon^{-1} m} Z_i > \epsilon \mid k_m \leq \epsilon^{-1} m \right) \leq \epsilon^{-1} m \mathbf{P}(m^{-1/2} Z_1 > \epsilon) \xrightarrow{m \rightarrow \infty} 0. \quad (2.17)$$

Combining (2.15), (2.16) and (2.17) establishes (2.13). \square

For each $m \geq \tau_1$, let $k[m]$ denote the unique natural number such that $\tau_{k[m]} \leq m < \tau_{k[m]+1}$, and for $m < \tau_1$ let $k[m] = 0$. By Corollary 2.9, the increments $(\tau_{k+1} - \tau_k, k \geq 1)$ are IID; since $\tau_{k+1} - \tau_k = \rho_{k+1} - \rho_k$, by Lemma 2.10 these increments have some positive finite expected value $\kappa \in (0, \infty)$. The law of large numbers then implies that $\tau_k/k \xrightarrow{\text{a.s.}} \kappa$ as $k \rightarrow \infty$, from which it follows easily that $m^{-1}k[m] \xrightarrow{\text{a.s.}} 1/\kappa$ as $m \rightarrow \infty$.

Proposition 2.13. *Fix any $x \in \mathbb{R}^{d \times N}$. As $m \rightarrow \infty$, we have*

$$m^{-1/2} \sup_{0 \leq t \leq m} |\bar{X}(t) - X_1(t)| \rightarrow 0 \quad (2.18)$$

in probability with respect to \mathbf{P}_x .

Proof. Since $\tau_1 < \infty$ almost surely with respect to \mathbf{P}_x by Lemma 2.10, it suffices to prove that as $m \rightarrow \infty$

$$m^{-1/2} \sup_{\tau_1 \leq t \leq m} |\bar{X}(t) - X_1(t)| \rightarrow 0$$

in probability with respect to \mathbf{P}_x . For all $t \geq 0$, we have

$$|\overline{X}(t) - X_1(t)| \leq \max_{j \in [N]} |X_j(t) - X_1(t)|.$$

Therefore

$$\sup_{\tau_1 \leq t \leq m} |\overline{X}(t) - X_1(t)| \leq \max_{i \in [k[m]]} \sup_{\tau_i \leq s \leq \tau_{i+1}} \max_{j \in [N]} |X_j(s) - X_1(s)|.$$

At time $\tau_i = T_{L_i} + 1$, the event $A_{T_{L_i}}$ has just occurred, which implies that $\max_{j \in [N]} |X_j(\tau_i) - X_1(\tau_i)| \leq 5 + 2^{-1}(N+1)^{-1} + 5(N-1)^{-1} < 11$. Moreover, the event $B_{T_{L_i}}$ is about to occur, which means that for all times $s \in [\tau_i, \rho_i] = [\tau_i, \tau_i + 1]$, every particle has distance at most $r_N = \frac{1}{4(N+1)}$ from the location of some time- τ_i particle. Therefore,

$$\sup_{\tau_i \leq s \leq \rho_i} \max_{j \in [N]} |X_j(s) - X_1(s)| \leq 12.$$

Writing

$$Z_i := \max \left(12, \sup_{\rho_i \leq s \leq \tau_{i+1}} \max_{j \in [N]} |X_j(s) - X_1(s)| \right)$$

for $i \geq 1$, it follows that

$$\sup_{\tau_1 \leq t \leq m} |\overline{X}(t) - X_1(t)| \leq \max_{i \in [k[m]]} Z_i.$$

To obtain (2.18), it thus suffices to verify the conditions of Fact 2.12 for $(Z_i)_{i \geq 1}$. Under \mathbf{P}_x , the random variables $\sup_{\rho_i \leq s \leq \tau_{i+1}} \max_{j \in [N]} |X_j(s) - X_1(s)|$ are identically distributed for $i \in [k[m]]$ by Proposition 2.8, from which it is immediate that $(Z_i)_{i \geq 1}$ are identically distributed. It only remains to show that $Z_1 \in L^2(\mathbf{P}_x)$. For this, observe that

$$\begin{aligned} & \max_{j \in [N]} |X_j(s) - X_1(s)|^2 \\ & \leq 4 \max_{j \in [N]} |X_j(s) - X_j(\tau_1)|^2 + 4 \max_{j \in [N]} |X_j(\tau_1) - X_1(s)|^2 \\ & \leq 4 \max_{j \in [N]} |X_j(s) - X_j(\tau_1)|^2 + 16 \max_{j \in [N]} |X_j(\tau_1) - X_1(\tau_1)|^2 + 16 |X_1(\tau_1) - X_1(s)|^2 \\ & \leq 20 \max_{j \in [N]} |X_j(s) - X_j(\tau_1)|^2 + 16 \max_{j \in [N]} |X_j(\tau_1) - X_1(\tau_1)|^2 \\ & \leq 20 \max_{j \in [N]} |X_j(s) - X_j(\tau_1)|^2 + 1600 \end{aligned}$$

where in the last inequality, we have used that $\max_{j \in [N]} |X_j(\tau_1) - X_1(\tau_1)|^2 \leq 10^2$ by the definitions of events $(A_{t,2})$ and $(A_{t,3})$. Taking the supremum over $s \in [\rho_1, \tau_2]$ and taking expectations, we infer that

$$\mathbf{E}_x[Z_1^2] \leq 144 + 20\mathbf{E}_x \left[\sup_{\rho_1 \leq s \leq \tau_2} \max_{j \in [N]} |X_j(s) - X_j(\tau_1)|^2 \right] + 1600$$

By the strong Markov property and (2.6), we obtain

$$\begin{aligned} \mathbf{E}_x \left[\sup_{\rho_1 \leq s \leq \tau_2} \max_{j \in [N]} |X_j(s) - X_j(\tau_1)|^2 \right] &= \mathbf{E}_\pi \left[\sup_{0 \leq s \leq \tau_1} \max_{j \in [N]} |X_j(s) - X_j(0)|^2 \right] \\ &\leq \mathbf{E}_\pi \left[\mathcal{J}_{\tau_1}^2 \sup_{0 \leq s \leq \tau_1} \max_{j \in [N]} |B_j(s)|^2 \right], \end{aligned}$$

which is finite, as seen in the proof of Proposition 2.11. This proves that $Z_1 \in L^2(\mathbf{P}_x)$; and establishes the convergence in (2.18). \square

2.5. Proof of Theorem 1.1

Set $\Delta_1 := X_1(\tau_2) - X_1(\tau_1)$, and denote by ν the law of Δ_1 under \mathbf{P}_x ; it is a Borel probability measure on \mathbb{R}^d defined by

$$\nu(B) := \mathbf{P}_x(\Delta_1^{-1}(B)) = \mathbf{P}_\pi(X_1(\tau_1) - X_1(0) \in B);$$

the second equality holds by Proposition 2.8 and makes it clear that the measure ν does not depend on the initial configuration $x \in \mathbb{R}^{d \times N}$. The rotational invariance of Brownian motion immediately implies the following lemma (whose proof we omit).

Lemma 2.14. *The measure ν is invariant under the action of the group $O(d, \mathbb{R})$ of d -by- d orthogonal matrices with entries in \mathbb{R} .*

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Fix $x \in \mathbb{R}^{d \times N}$. We work on the probability space $(\Omega, \mathcal{F}, \mathbf{P}_x)$. When $N = 1$, the process \mathbf{X} consists of a single Brownian motion so the theorem holds with $\sigma(d, 1) = 1$ and no drift term.

Next suppose that $N = 2$. In this case, the first construction of the BBB process given in Section 1.1 simplifies as follows. Let $(\zeta_i; i \geq 1)$ be the jump times for a counting process $\mathcal{J} = (\mathcal{J}_t, t \geq 0)$ for a Poisson point process on $[0, \infty)$ with rate 2 and define $\zeta_0 = 0$. We denote by $\mathbf{X}_1 = (X_1(t), t \geq 0)$ and $\mathbf{B} = (B(t), t \geq 0)$ two d -dimensional Brownian motions started from $x_1 \in \mathbb{R}^d$ and $0 \in \mathbb{R}^d$, respectively. Define a process $\mathbf{X}_2 = (X_2(t), t \geq 0)$ by $X_2(0) = x_2 \in \mathbb{R}^d$, and inductively for each $i \geq 0$ by

$$X_2(t) = X_1(\zeta_i) + B(t) - B(\zeta_i), \quad t \in [\zeta_i, \zeta_{i+1}).$$

Thus, when $N = 2$, the BBB process $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$ consists of a Brownian motion \mathbf{X}_1 to which we attach branches \mathbf{X}_2 that can only grow between consecutive branching times. For $s \geq 0$, let $k[s]$ be the unique positive integer such that $\zeta_{k[s]} \leq s \leq \zeta_{k[s]+1}$. By the triangle inequality, we have

$$\begin{aligned} & \sup_{0 \leq t \leq 1} \left| \frac{\bar{X}(tm)}{\sqrt{m}} - \frac{X_1(tm)}{\sqrt{m}} \right| \\ &= \sup_{0 \leq t \leq 1} \left| \frac{X_2(tm)}{2\sqrt{m}} - \frac{X_1(tm)}{2\sqrt{m}} \right| \\ &\leq \sup_{0 \leq t \leq 1} \left| \frac{X_2(tm)}{2\sqrt{m}} - \frac{X_2(\zeta_{k[tm]})}{2\sqrt{m}} \right| + \sup_{0 \leq t \leq 1} \left| \frac{X_1(tm)}{2\sqrt{m}} - \frac{X_1(\zeta_{k[tm]})}{2\sqrt{m}} \right| \\ &\leq \max_{0 \leq i < k[m]} \sup_{s \in [\zeta_i, \zeta_{i+1}]} \frac{|X_2(s) - X_2(\zeta_i)|}{2\sqrt{m}} + \max_{0 \leq i < k[m]} \sup_{s \in [\zeta_i, \zeta_{i+1}]} \frac{|X_1(s) - X_1(\zeta_i)|}{2\sqrt{m}}. \end{aligned}$$

We claim that the last expression converges to 0 in probability. Since $\zeta_2 - \zeta_1$ has exponential tails, and for each $j \in \{1, 2\}$,

$$(X_j(s) - X_j(\zeta_1), s \in [\zeta_1, \zeta_2])$$

is a Brownian motion, we infer that

$$\sup_{s \in [\zeta_1, \zeta_2]} |X_j(s) - X_j(\zeta_1)| \in L^2(\mathbf{P}_x).$$

Therefore, Fact 2.12 implies the convergence to 0 in probability. Since the initial value of \bar{X} is sent to 0 when $m \rightarrow \infty$, we conclude that \bar{X} satisfies an invariance principle with $\sigma(d, 2) = 1$.

Suppose now that $N \geq 3$. By Lemma 2.7 and Proposition 1.4 the barycenter \bar{X} satisfies an invariance principle, i.e. there exists a non-zero d -by- d matrix $\Sigma = \Sigma(d, N)$ such that (1.3) holds. We next show that the drift term in the invariance principle is zero for any $d \geq 1$ and $N \geq 3$. By Proposition 1.4, the drift is $\alpha = \mathbf{E}_x[\Delta_1] \mathbf{E}_x[\tau_2 - \tau_1]^{-1}$. Moreover, we note that property (4) in Proposition 1.4, combined with the Cauchy-Schwarz inequality, immediately implies that $\Delta_i = X_1(\tau_{i+1}) - X_1(\tau_i) \in L^1(\mathbf{P}_\pi)$, for each $i \geq 1$. In particular, $\mathbf{E}_x[\Delta_1] < \infty$. The rotational

invariance from Lemma 2.14 then immediately implies that $\mathbf{E}_x[\Delta_1] = 0$, and hence $\alpha = 0$. Finally, since for any $m > 0$, the process $(m^{-1/2}\bar{X}(tm), 0 \leq t \leq 1)$ is rotationally invariant, then the limit process $(\Sigma B(t), 0 \leq t \leq 1)$ is also rotationally invariant. Considering rotations of the form $(x, y) \mapsto (-y, x)$ applied to all pairs of coordinates, rotational invariance implies that $(\Sigma B(t), 0 \leq t \leq 1)$ has uncorrelated and identically distributed coordinates. Since $(\Sigma B(t), 0 \leq t \leq 1)$ is a Gaussian process, this implies its coordinates are in fact IID, and thus $\Sigma(d, N) = \sigma(d, N)I_d$ for some $\sigma(d, N) > 0$. \square

3. Proofs of Technical Ingredients

In this section, we include the proofs of Lemma 2.4, Lemma 2.5, and Theorem 1.3. Recall from Section 2.1 that $C_L = \{x \in \mathbb{R}^{d \times N} : E(x) \leq L\}$ is the set of configurations with extent at most L .

3.1. Two events with positive probability

We begin by proving that the events A and B defined by $(A_{t,1})$ - $(A_{t,3})$ and $(B_{t,1})$, $(B_{t,2})$ have positive probability.

Lemma 3.1. *Let $A = A_0 = A_{0,1} \cap A_{0,2} \cap A_{0,3}$ be defined by $(A_{t,1})$ - $(A_{t,3})$ with $t = 0$. There exists $\epsilon = \epsilon_{L,N} > 0$ such that*

$$\inf_{x \in C_L} \mathbf{P}_x(A) > \epsilon.$$

Proof. First, note that the event A can be rewritten for the process X (instead of $X - \bar{X}(0)$) by assuming that $\bar{X}(0) = 0$. Hence, it is enough to show that there exists $\epsilon = \epsilon_{L,N} > 0$ such that

$$\inf\{\mathbf{P}_x(A) : x \in C_L, \bar{x} = 0\} > \epsilon.$$

Fix any $x \in C_L$ with $\bar{x} = 0$. We write A as

$$A = A_{0,1} \cap E_1 \cap E_2 \cap E_3,$$

where the events E_i are defined as follows, using the same notation as in $(A_{t,1})$ - $(A_{t,3})$. Define

$$\begin{aligned} E_1 &:= \{X_j(1) \in B(-5e_1, r_N), \forall j \in G'\}, \\ E_2 &:= \{X_j(1) \in B(5e_1, r_N), \forall j \in D'\}, \end{aligned}$$

and

$$E_3 := \{X_1(1) \in B(\gamma, r_N)\},$$

where as before $r_N := \frac{1}{4(N+1)} \in (0, 1)$, and we recall that γ is roughly the location of the barycenter of $G' \cup D'$. We then have

$$\begin{aligned} \mathbf{P}_x(A) &= \mathbf{P}_x(E_1 \cap E_2 \cap E_3 | A_{0,1}) \mathbf{P}_x(A_{0,1}) \\ &= \prod_{k=1}^3 \mathbf{P}_x \left(E_k | A_{0,1}, \bigcap_{j=1}^{k-1} E_j \right) \mathbf{P}_x(A_{0,1}). \end{aligned}$$

Since we start from a deterministic configuration x , and conditioning on $A_{0,1}$ means there are no branching events between times 0 and 1, the positions of the particles at time 1 are independent given $A_{0,1}$. Therefore we can bound each term in the last product as follows: writing $\mathcal{N}(x, I_d)$ for a d -dimensional Gaussian centered at x ,

$$\begin{aligned} \mathbf{P}_x(E_1 | A_{0,1}) &\geq \left(\min_{j \in G'} \mathbf{P}(\mathcal{N}(x_j, I_d) \in B(-5e_1, r_N)) \right)^{|G'|}, \\ \mathbf{P}_x(E_2 | A_{0,1}, E_1) &\geq \left(\min_{j \in D'} \mathbf{P}(\mathcal{N}(x_j, I_d) \in B(5e_1, r_N)) \right)^{|D'|}, \\ \mathbf{P}_x(E_3 | A_{0,1}, E_1 \cap E_2) &\geq \mathbf{P}(\mathcal{N}(x_1, I_d) \in B(\gamma, r_N)), \end{aligned}$$

and $\mathbf{P}_x(A_{0,1}) = e^{-N}$. Taking the infimum over admissible configurations $x \in C_L$ satisfying $\bar{x} = 0$, we obtain

$$\begin{aligned} & \inf\{\mathbf{P}_x(A) : x \in C_L, \bar{x} = 0\} \\ & \geq e^{-N} \inf_{x \in C_L, \bar{x} = 0} \left[\left(\min_{j \in G'} \mathbf{P}(\mathcal{N}(x_j, I_d) \in B(-5e_1, r_N)) \right)^{\lfloor \frac{N-1}{2} \rfloor} \times \right. \\ & \quad \left. \times \left(\min_{j \in D'} \mathbf{P}(\mathcal{N}(x_j, I_d) \in B(5e_1, r_N)) \right)^{\lceil \frac{N-1}{2} \rceil} \mathbf{P}(\mathcal{N}(x_1, I_d) \in B(\gamma, r_N)) \right] \\ & =: \epsilon_{L,N} > 0. \quad \square \end{aligned}$$

Lemma 3.2. *Let $B = B_0 = B_{0,1} \cap B_{0,2}$ be defined by $(B_{t,1}), (B_{t,2})$. There exists $\epsilon = \epsilon_{L,N} > 0$ such that*

$$\inf_{x \in C_L} \mathbf{P}_x(B|A) > \epsilon.$$

Proof. Let $\mathcal{S}' \subset \mathcal{S}$ be the set of possible values of X at time 1, given that A occurred. Namely, $x \in \mathcal{S}'$ if and only if

- (a) $C = \{1\}$, $G = \{2, \dots, \lceil \frac{N+1}{2} \rceil\}$, and $D = \{\lceil \frac{N+1}{2} \rceil + 1, \dots, N\}$,
- (b) $x_1 \in B(\gamma, r_N)$,
- (c) $x_j \in B(-5e_1, r_N)$ for all $j \in G$, and $x_j \in B(5e_1, r_N)$ for all $j \in D$.

In particular, we have that $G = G'$ and $D = D'$. By the Markov property and $(A_{t,1})$ - $(A_{t,3})$, we have

$$\inf_{x \in C_L} \mathbf{P}_x(B|A) \geq \inf_{x \in \mathcal{S}'} \mathbf{P}_x(\tilde{B}), \quad (3.1)$$

where $\tilde{B} = \tilde{B}_{0,1} \cap \tilde{B}_{0,2}$ is the event B “taking place in the time interval $[0, 1]$ ” (rather than $[1, 2]$). We can decompose \tilde{B} as the intersection of the following five events:

$$U_1 := \bigcap_{j \in G' \cup D'} \{X_j \text{ doesn't branch from time 0 until time 1 or until it is killed, whichever comes first}\} \subset \tilde{B}_{0,1},$$

$$U_2 := \{\text{Each particle in } G' \text{ stays in } B(-5e_1, 2r_N) \text{ from time 0 until time 1 or until it is killed, whichever comes first}\} \subset \tilde{B}_{0,2},$$

$$U_3 := \{\text{Each particle in } D' \text{ stays in } B(5e_1, 2r_N) \text{ from time 0 until time 1 or until it is killed, whichever comes first}\} \subset \tilde{B}_{0,2},$$

$$U_4 := \{\text{The descendants of } X_1 \text{ stay in } B(\gamma, 2r_N) \text{ for all times in } [0, 1]\} \subset \tilde{B}_{0,2},$$

$$U_5 := \{X_1 \text{ and its descendants branch at least } N - 1 \text{ times in } [0, 1]\} \subset \tilde{B}_{0,1}.$$

Observe that for any $x \in \mathcal{S}'$, we have

$$\mathbf{P}_x(\tilde{B}) = \prod_{k=1}^5 \mathbf{P}_x(U_k | \bigcap_{j=1}^{k-1} U_j). \quad (3.2)$$

We bound each factor in (3.2) individually as follows. We have

$$\mathbf{P}_x(U_1) \geq \mathbf{P}_x(\text{a particle doesn't branch in } [0, 1])^{N-1} = e^{-(N-1)},$$

Given U_1 , the particles in G' are independent until time 1 or until they are killed, whichever comes first. Therefore, if $B = (B(s), s \geq 0)$ denotes a d -dimensional Brownian motion, then we have

$$\mathbf{P}_x(U_2|U_1) \geq \min_{j \in G'} \mathbf{P}(B(s) \in B(-5e_1, 2r_N), \forall s \in [0, 1] | B(0) = x_j)^N.$$

Similarly, we have

$$\mathbf{P}_x(U_3|U_1 \cap U_2) \geq \min_{j \in D'} \mathbf{P}(B(s) \in B(5e_1, 2r_N), \forall s \in [0, 1] | B(0) = x_j)^N.$$

Event U_4 contains the event that particle X_1 does not branch in $[0, 1]$, and that its trajectory stays in $B(\gamma, 2r_N)$ for all times in $[0, 1]$. Thus, if $B = (B(s), s \geq 0)$ is a d -dimensional Brownian motion, we have

$$\mathbf{P}_x(U_4|U_1 \cap U_2 \cap U_3) \geq e^{-1} \mathbf{P}(B(s) \in B(\gamma, 2r_N), \forall s \in [0, 1] | B(0) = x_1).$$

Event U_5 contains the event that particle X_1 branches $N - 1$ times in the time interval $[0, 1]$, and no other particle in the BBB branches in the time interval $[0, 1]$. Thus

$$\mathbf{P}_x(U_5|U_1 \cap U_2 \cap U_3 \cap U_4) \geq e^{-(N-1)} \frac{(1 - e^{-1})^{N-1}}{(N-1)!}.$$

Combining these lower bounds, and taking the infimum over $x \in \mathcal{S}'$ in (3.2), we obtain

$$\inf_{x \in C_L} \mathbf{P}_x(B|K) > \epsilon_{L,N},$$

for some $\epsilon_{L,N} > 0$. The lemma then follows from (3.1). \square

Proof of Lemma 2.4. Lemmas 3.1 and 3.2 imply that

$$\inf_{x \in C_L} \mathbf{P}_x(A \cap B) \geq \left(\inf_{x \in C_L} \mathbf{P}_x(A) \right) \left(\inf_{x \in C_L} \mathbf{P}_x(B|A) \right) > 0. \quad \square$$

3.2. Proof of Lemma 1.6

This section is devoted to proving the deterministic Lemma 1.6. For this we will use the following two claims.

Claim 3.3. *Fix a configuration x, w and an integer $\ell \in [N]$ with $w_\ell > 0$. Write $k = k(x, w, \ell)$ and $w^* = g(x, w, \ell)$. If $w_k^* \geq 1$ and $k \neq \ell$ then also $k(x, w^*, \ell) \neq \ell$.*

Proof. Let

$$y = \frac{1}{N - w_\ell - w_k} \sum_{i \in [N] \setminus \{k, \ell\}} w_i x_i,$$

and define

$$b = \frac{N - w_\ell - w_k}{N + 1} y + \frac{(w_\ell + 1)}{N + 1} x_\ell + \frac{w_k}{N + 1} x_k;$$

this is the barycenter of the configuration obtained from x, w after letting x_ℓ branch (increasing its weight by 1) but before any killing takes place (so the total weight is $N + 1$ rather than N). Likewise, upon killing particle x_k and letting x_ℓ branch consecutively, define

$$b^* = \frac{N - w_\ell - w_k}{N + 1} y + \frac{(w_\ell^* + 1)}{N + 1} x_\ell + \frac{w_k^*}{N + 1} x_k.$$

We have $(N - w_\ell - w_k)y = \sum_{i \in [N] \setminus \{k, \ell\}} w_i x_i = \sum_{i \in [N] \setminus \{k, \ell\}} w_i^* x_i$ since $w_i^* = w_i$ for all $i \in [N] \setminus \{k, \ell\}$ and $N - w_\ell^* - w_k^* = N - w_\ell - w_k$ since $w_\ell^* = w_\ell + 1$ and $w_k^* = w_k - 1$. Thus, the point b^* is the barycenter of the configuration obtained from x, w^* by letting x_ℓ branch but before any killing takes place, and also,

$$b^* = \frac{N - w_\ell - w_k}{N + 1} y + \frac{(w_\ell + 2)}{N + 1} x_\ell + \frac{(w_k - 1)}{N + 1} x_k = b + \frac{x_\ell - x_k}{N + 1}. \quad (3.3)$$

As depicted in Figure 3 below, since $|x_\ell - b| < |x_k - b|$, it follows immediately that $|x_\ell - b^*| < |x_k - b^*|$.

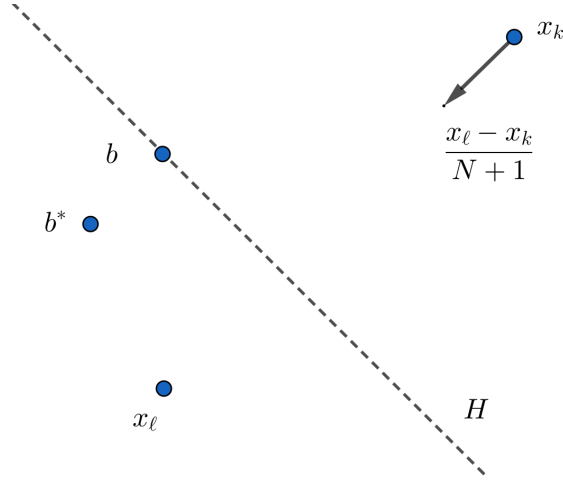


Figure 3. Pictorial description of (3.3) showing that $|x_\ell - b^*| < |x_k - b^*|$. Here, H is the hyperplane perpendicular to $x_\ell - x_k$ and passing through b .

Since $w_k^* \geq 1$, this means there is at least one positive-weight particle further from b^* than x_ℓ , and thus $k(x, w^*, \ell) \neq \ell$. \square

Claim 3.4. Fix a configuration x, w , let $b_0 = N^{-1} \sum_{i=1}^N w_i x_i$, and let

$$\ell := \operatorname{argmin}_{i \in [N]: w_i > 0} |x_i - b_0|. \quad (3.4)$$

If there exists $j \in [N] \setminus \{\ell\}$ with $w_j > 0$ then $k(x, w, \ell) \neq \ell$.

Proof. Since the killing rules for deterministic configurations are invariant under affine transformations, we may assume that $b_0 = (0, 0, \dots, 0)$ and $x_\ell = (1, 0, \dots, 0)$. Writing

$$z = \frac{1}{N - w_\ell} \sum_{j \in [N] \setminus \{\ell\}} w_j x_j,$$

we have

$$b_0 = \frac{N - w_\ell}{N} z + \frac{w_\ell}{N} x_\ell,$$

so $z = (N b_0 - w_\ell x_\ell) / (N - w_\ell) = (-w_\ell / (N - w_\ell), 0, \dots, 0)$. Since z is the (weighted) barycenter of the particles excluding x_ℓ , it follows that there is $j \neq \ell$ with $w_j > 0$ such that x_j is an element of the halfspace $\mathcal{P} = \{(p_1, \dots, p_d) : p_1 \leq -w_\ell / (N - w_\ell)\}$.

Now, similar to the proof of Claim 3.3, write

$$b = \frac{N - w_\ell}{N + 1} z + \frac{w_\ell + 1}{N + 1} x_\ell$$

for the barycenter of the configuration obtained from x, w after letting x_ℓ branch but before any killing takes place. An easy calculation gives that $b = (1/(N + 1), 0, \dots, 0)$, so $|x_\ell - b| < |x_\ell - b_0|$. Moreover, for a particle $x_j \in \mathcal{P}$ as in the previous paragraph, we have $|x_j - b| > |x_j - b_0|$. By our choice of ℓ we also have $|x_\ell - b_0| < |x_j - b_0|$, and thus $|x_\ell - b| < |x_j - b|$. Thus there is at least one particle further from b than x_ℓ , so $k(x, w, \ell) \neq \ell$. \square

We now conclude the proof as follows.

Proof of Lemma 1.6. Fix any unambiguous configuration x, w (recall (1.4)) with at least two particles with non-zero weights. For b_0 as in Claim 3.4, let $\ell = \operatorname{argmin}_{i \in [N]: w_i > 0} |x_i - b_0|$. Let $w^{(0)} = w$, and for $j \geq 0$, $w^{(j+1)} = g(x, w^{(j)}, \ell)$, and $k_j = k(x, w^{(j)}, \ell)$. By Claim 3.4, $k_0 \neq \ell$. By Claim 3.3 and induction, for all $j \geq 0$ such that $w_{k_j}^{(j+1)} \geq 1$, we have $k(x, w^{(j+1)}, \ell) \neq \ell$, so

$w_\ell^{(j+1)} = w_\ell^{(j)} + 1$. Since the maximum possible weight is N , it follows that there is $j \leq N - 1$ such that $w_{k_j}^{(j+1)} = 0$.

We have shown that, starting from any unambiguous configuration with at least two particles of non-zero weight, there exists a sequence of at most $N - 1$ branching events which reduces the number of particles with nonzero weight by at least one. Repeating such a procedure at most $N - 1$ times, we necessarily obtain a configuration with at most one nonzero weight. This completes the proof. \square

Given $\delta > 0$, let $x \in \mathbb{R}^{d \times N}$ be a δ -unambiguous configuration (as defined by (1.5)) and introduce the notation

$$\mathbb{R}_{\delta/4, x}^{d \times N} := \left\{ y \in \mathbb{R}^{d \times N} : y = (y_1, \dots, y_N) \in \prod_{i=1}^N B\left(x_i, \frac{\delta}{4}\right) \right\}, \quad (3.5)$$

which is the set of generic configurations within $\frac{\delta}{4}$ of x . The following corollary will be essential to the proof of Lemma 2.5 in the next section (Section 3.3).

Corollary 3.5. *Given any δ -unambiguous configuration x, w , Lemma 1.6 assigns the same collapsing sequence of branching events to y, w for all $y \in \mathbb{R}_{\delta/4, x}^{d \times N}$.*

Proof. It is immediate from the definition of $\mathbb{R}_{\delta/4, x}^{d \times N}$ and the fact that x is δ -unambiguous that any element $y \in \mathbb{R}_{\delta/4, x}^{d \times N}$ is $\frac{\delta}{2}$ -unambiguous. Moreover, $y_i \in B(x_i, \frac{\delta}{4})$ for each $i \in [N]$ so at any step in the proof of Lemma 1.6, the barycenter of y, w can only be at most $\frac{\delta}{4}$ away from the barycenter of x, w . Hence, Lemma 1.6 assigns the same sequence to y, w as to x, w . \square

3.3. Harris recurrence for the recentered BBB

In this section we prove Lemma 1.5, Lemma 2.5, and finally Theorem 1.3, which asserts the Harris recurrence of $X - \bar{X}$.

Recall that for $\delta > 0$, we defined $\xi_\delta := \inf\{t \geq 0 : X(t) \text{ is } \delta\text{-unambiguous}\}$, where the meaning of δ -unambiguous is given by (1.5). Lemma 1.5 states that for δ small, the event $\{\xi_\delta < 1\}$ occurs with probability as close to 1 as we wish, uniformly over starting configurations.

Proof of Lemma 1.5. For each $\alpha \in (0, 1)$, define the event

$$D_\alpha := \{\text{no branching occurs in } [0, \alpha]\}.$$

For each $t \geq 0$ and $j \in [N]$, we define

$$r_j(t) = |X_j(t) - \bar{X}(t)|.$$

For any $x \in \mathbb{R}^{d \times N}$ and any $\delta > 0$, we have

$$\begin{aligned} \mathbf{P}_x(\xi_\delta \geq 1 | D_\alpha) &= \mathbf{P}_x \left(\bigcap_{0 \leq t < 1} \left\{ \min_{i \neq j \in [N]} |r_i(t) - r_j(t)| \leq \delta \right\} \middle| D_\alpha \right) \\ &\leq \mathbf{P}_x \left(\min_{i \neq j \in [N]} \left| r_i\left(\frac{\alpha}{2}\right) - r_j\left(\frac{\alpha}{2}\right) \right| \leq \delta \middle| D_\alpha \right) \\ &\leq \sum_{i \neq j \in [N]} \mathbf{P}_x \left(\left| r_i\left(\frac{\alpha}{2}\right) - r_j\left(\frac{\alpha}{2}\right) \right| \leq \delta \middle| D_\alpha \right) \\ &\leq \binom{N}{2} \max_{i \neq j \in [N]} \mathbf{P}_x \left(\left| r_i\left(\frac{\alpha}{2}\right) - r_j\left(\frac{\alpha}{2}\right) \right| \leq \delta \middle| D_\alpha \right). \end{aligned}$$

Moreover,

$$\mathbf{P}_x \left(\left| r_i\left(\frac{\alpha}{2}\right) - r_j\left(\frac{\alpha}{2}\right) \right| \leq \delta \middle| D_\alpha \right) = \mathbf{P}_x \left(\left| g_{i,j} \left(X\left(\frac{\alpha}{2}\right) \right) - g_{j,i} \left(X\left(\frac{\alpha}{2}\right) \right) \right| \leq \delta \middle| D_\alpha \right), \quad (3.6)$$

where

$$g_{i,j} \left(X \left(\frac{\alpha}{2} \right) \right) := \left| \frac{N-1}{N} X_i \left(\frac{\alpha}{2} \right) - \frac{1}{N} X_j \left(\frac{\alpha}{2} \right) - \frac{1}{N} \sum_{k \in [N] \setminus \{i,j\}} X_k \left(\frac{\alpha}{2} \right) \right|.$$

is the distance between X_i and \bar{X} at time $\frac{\alpha}{2}$. Thus, by the tower law, we can write (3.6) as

$$\mathbf{E}_x \left[\mathbf{P}_x \left(\left| g_{i,j} \left(X \left(\frac{\alpha}{2} \right) \right) - g_{j,i} \left(X \left(\frac{\alpha}{2} \right) \right) \right| \leq \delta \mid \sigma \left((X_k(s))_{k \in [N] \setminus \{i,j\}} : 0 \leq s \leq \frac{\alpha}{2} \right), D_\alpha \right) \mid D_\alpha \right].$$

Given D_α , the particle positions at time $\frac{\alpha}{2}$ are independent Gaussians. So, as $\delta \downarrow 0$, the event in the conditional probability then requires that the norms of two independent Gaussians be arbitrarily close at time $\frac{\alpha}{2}$. It follows that

$$\begin{aligned} & \lim_{\delta \downarrow 0} \sup_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x(\xi_\delta \geq 1 \mid D_\alpha) \\ &= \lim_{\delta \downarrow 0} \sup_{x \in \mathbb{R}^{d \times N}} \max_{i \neq j \in [N]} \mathbf{P}_x \left(\left| r_i \left(\frac{\alpha}{2} \right) - r_j \left(\frac{\alpha}{2} \right) \right| \leq \delta \mid D_\alpha \right) \\ &= 0. \end{aligned} \tag{3.7}$$

Finally, we have

$$\begin{aligned} \mathbf{P}_x(\xi_\delta \geq 1) &= \mathbf{P}_x(\xi_\delta \geq 1, D_\alpha) + \mathbf{P}_x(\xi_\delta \geq 1, D_\alpha^c) \\ &= e^{-\alpha N} \mathbf{P}_x(\xi_\delta \geq 1 \mid D_\alpha) + (1 - e^{-\alpha N}) \mathbf{P}_x(\xi_\delta \geq 1 \mid D_\alpha^c) \\ &\leq e^{-\alpha N} \mathbf{P}_x(\xi_\delta \geq 1 \mid D_\alpha) + 1 - e^{-\alpha N}. \end{aligned}$$

Thus taking the supremum over $x \in \mathbb{R}^{d \times N}$, and subsequently the limits as $\delta \downarrow 0$ and as $\alpha \downarrow 0$, and using (3.7), we deduce that

$$\lim_{\delta \downarrow 0} \sup_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x(\xi_\delta \geq 1) = 0.$$

We conclude that for any $\epsilon > 0$, there exists $\delta = \delta_\epsilon > 0$ such that

$$\inf_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x(\xi_\delta < 1) > 1 - \epsilon.$$

□

We now turn to the proof of Lemma 2.5. Recall that the time- t extent of the process is defined as

$$E(X(t)) := \max_{i \neq j \in [N]} |X_i(t) - X_j(t)|;$$

the lemma asserts that the extent reaches bounded values in a bounded time with high probability. For the proof we need one final definition. For any $t \geq s \geq 0$ and $\delta > 0$, let

$$H_\delta(s, t] := \{ \text{Each particle stays within distance } \delta \text{ of the location of its time-}s \text{ ancestor from time } s \text{ until it is killed or until time } t, \text{ whichever comes first} \}.$$

Note that for any $x \in \mathbb{R}^{d \times N}$,

$$\mathbf{P}_x(H_\delta(s, t] \mid \mathcal{F}_s) \geq e^{-N(t-s)} \mathbf{P} \left(\sup_{r \in [0, t-s]} |B(r)| < \delta \right)^N > 0,$$

where $(B(r), r \geq 0)$ is a standard d -dimensional Brownian motion starting from the origin. This holds since one way for $H_\delta(s, t]$ to occur is for no branching to occur in (s, t) and for no particle to travel distance $\geq \delta$.

Proof of Lemma 2.5. We give a two-step argument. Throughout the proof we assume that $L \geq 2$.

Step 1: *Getting to a δ -generic configuration confined to small balls of radius $\frac{\delta}{4}$ with positive probability.* Fix $\epsilon \in (0, 1)$. By Lemma 1.5, there exists $\delta = \delta_\epsilon > 0$ such that

$$1 - \epsilon < \inf_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x(\xi_\delta < 1).$$

Moreover, there exists $c = c_{L, \delta} > 0$ such that

$$\inf_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x(H_{\delta/4}(\xi_\delta, L] | \xi_\delta < 1) > c.$$

It follows that

$$\inf_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x(H_{\delta/4}(\xi_\delta, L], \xi_\delta < 1) > (1 - \epsilon)c > 0. \quad (3.8)$$

For any $x \in \mathbb{R}^{d \times N}$, we have

$$\mathbf{P}_x(T^{(L)} > L) = \mathbf{P}_x(T^{(L)} > L | H_{\delta/4}(\xi_\delta, L], \xi_\delta < 1) \mathbf{P}_x(H_{\delta/4}(\xi_\delta, L], \xi_\delta < 1),$$

so by (3.8) the result follows if we can show $\lim_{L \rightarrow \infty} \mathbf{P}_x(T^{(L)} > L | H_{\delta/4}(\xi_\delta, L], \xi_\delta < 1) = 0$.

Step 2: *Application of the deterministic lemma (Lemma 1.6) and of Corollary 3.5.*

By Lemma 1.6, there exists a deterministic sequence $(i_j)_{j=1}^m$ of $m \leq (N-1)^2$ branching events that collapses the configuration $X(\xi_\delta), (1, 1, \dots, 1)$. For convenience, introduce the notation

$$\nu_x(\cdot) := \mathbf{P}_x(\cdot | H_{\delta/4}(\xi_\delta, L], \xi_\delta < 1)$$

and, recalling the definition of $\mathbb{R}_{\delta, x}^{d \times N}$ from (3.5),

$$E_L := \{x \in \mathbb{R}_{\delta/4, X(\xi_\delta)}^{d \times N} : E(x) > L\}.$$

By the law of total probability and the Markov Property, we obtain

$$\begin{aligned} & \nu_x(T^{(L)} > L) \\ &= \nu_x(T^{(L)} > L - 2) \nu_x(T^{(L)} > L | T^{(L)} > L - 2) \\ &= \nu_x(T^{(L)} > L - 2) \int_{E_L} \nu_x(T^{(L)} > L | T^{(L)} > L - 2, X(L - 2) = y) \nu_x(X(L - 2) \in dy) \\ &\leq \sup_{x \in \mathbb{R}^{d \times N}} \nu_x(T^{(L)} > L - 2) \sup_{y \in E_L} \nu_y(T^{(L)} > 2). \end{aligned} \quad (3.9)$$

Thus, it suffices to show that there exists $C \in (0, 1)$ such that

$$\sup_{x \in E_L} \nu_x(T^{(L)} > 2) < C. \quad (3.10)$$

Consider the event

$$F := \{(i_j)_{j=1}^m \text{ are the first } m \text{ branching events to occur in } (\xi_\delta, 2]\}.$$

We have the containment $F \subset \{T^{(L)} \leq 2\}$, so taking complements

$$\{T^{(L)} > 2\} \subset F^c = \{(i_j)_{j=1}^m \text{ are not the first } m \text{ branching events of } (\xi_\delta, 2]\}.$$

Using the first construction of the BBB process from Section 1.1 and the fact that the same sequence $(i_j)_{j=1}^m$ collapses any configuration $x \in E_L$ by Corollary 3.5, we obtain that

$$\begin{aligned} \nu_x(T^{(L)} > 2) &\leq \nu_x(F^c) \\ &= 1 - \nu_x(\mathcal{J}(2) - \mathcal{J}(\xi_\delta) \geq m) \nu_x(F^c | \mathcal{J}(2) - \mathcal{J}(\xi_\delta) \geq m) \\ &\leq 1 - \mathbf{P}(\text{Poisson}(N) \geq m) \cdot (1 - N^{-m}) \\ &= C < 1 \end{aligned}$$

for all $x \in E_L$, which proves (3.10). It then follows from (3.9) that

$$\lim_{L \rightarrow \infty} \sup_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x(T^{(L)} > L) \leq \lim_{L \rightarrow \infty} \sup_{x \in \mathbb{R}^{d \times N}} \nu_x(T^{(L)} > L) \leq \lim_{L \rightarrow \infty} C^{\lfloor L/2 \rfloor} = 0. \quad \square$$

We will now prove Theorem 1.3 using Lemma 2.5. We use Φ to denote the recentered BBB process, so that $\Phi_t := X(t) - \bar{X}(t)$ for all $t \geq 0$. For each $t > 0$ and $x \in \mathbb{R}^{d \times N}$, let $\mu_{t,x} := \mathbf{P}_x \circ \Phi_t^{-1}$ be the law of Φ_t given that $\Phi_0 = x - \bar{x}$. Denote by φ the N -fold product measure of a d -dimensional standard Gaussian and recall the notation

$$C_L := \{x \in \mathbb{R}^{d \times N} : E(x) \leq L\}$$

We begin with the following key estimate.

Lemma 3.6. *For any $L \geq 0$, there exists $\gamma = \gamma(L, N) > 0$ such that*

$$\inf_{x \in C_L} \inf_{1 \leq t \leq 2} \mu_{t,x}(A) \geq \gamma \varphi(A), \quad \text{for all Borel } A \subset \mathbb{R}^{d \times N} \quad (3.11)$$

Proof. Let $L \geq 0$. Define

$$D := \{\text{no branching events occur in } [0, 2]\}.$$

By monotonicity and independence, for any Borel $A \subset \mathbb{R}^{d \times N}$, we obtain

$$\begin{aligned} \mu_{t,x}(A) &\geq \mu_{t,x}(A \cap D) \\ &= \mu_{t,x}(A|D) \mu_{t,x}(D) \\ &= e^{-2N} \int_A \prod_{i=1}^N \frac{e^{-\frac{|y_i - (x_i - \bar{x})|^2}{2t}}}{(2\pi t)^{d/2}} dy_1 \cdots dy_N. \end{aligned}$$

Moreover, for each $x \in C_L$, $y \in \mathbb{R}^{d \times N}$, and $t \in [1, 2]$,

$$\frac{e^{-\frac{|y_i - (x_i - \bar{x})|^2}{2t}}}{(2\pi t)^{d/2}} \geq \frac{e^{-\frac{|y_i|^2}{2t}} e^{-\frac{|x_i - \bar{x}|^2}{2t}}}{(2\pi t)^{d/2}} \geq \frac{e^{-\frac{L^2}{2}} e^{-\frac{|y_i|^2}{2}}}{2^{d/2} (2\pi)^{d/2}},$$

which implies that for each $x \in C_L$ and $t \in [1, 2]$,

$$\begin{aligned} e^{-2N} \int_A \prod_{i=1}^N \frac{e^{-\frac{|y_i - (x_i - \bar{x})|^2}{2t}}}{(2\pi t)^{d/2}} dy_1 \cdots dy_N &\geq e^{-2N} \frac{e^{-\frac{NL^2}{2}}}{2^{dN/2}} \frac{1}{(2\pi)^{dN/2}} \int_A \prod_{i=1}^N e^{-\frac{|y_i|^2}{2}} dy_1 \cdots dy_N \\ &= e^{-2N} \frac{e^{-\frac{NL^2}{2}}}{2^{Nd/2}} \varphi(A). \end{aligned}$$

Thus (3.11) holds if we take $\gamma(L, N) := \frac{e^{-2N} e^{-\frac{NL^2}{2}}}{2^{Nd/2}} > 0$. \square

Proof of Theorem 1.3. We wish to show that Φ is Harris recurrent. According to Definition 1.2, it suffices to show that for the same choice of φ as above, we have for all $A \subset \mathbb{R}^{d \times N}$ Borel,

$$\varphi(A) > 0 \Rightarrow \inf_{x \in \mathbb{R}^{d \times N}} \mathbf{P}_x(\eta_A = \infty) = 1,$$

where $\eta_A = \int_0^\infty \mathbb{1}_{\{\Phi_t \in A\}} dt$. By Lemma 2.5 and Remark 2.6 (i), the stopping times $(T_i, i \geq 1)$ defined by (2.1) are finite \mathbf{P}_x -almost surely. Fix $L \geq 0$ and any Borel set $A \subset \mathbb{R}^{d \times N}$. For each integer $i \geq 1$, define

$$\eta_A^i := \int_{T_i+1}^{T_{i+2}} \mathbb{1}_{\{\Phi_t \in A\}} dt.$$

For all $i \geq 1$, the strong Markov property, Tonelli's Theorem and Lemma 3.6 imply that

$$\begin{aligned} \mathbf{E}_x[\eta_A^i] &= \mathbf{E}_x \left[\mathbf{E}_{X(T_i)}[\eta_A^1] \right] \\ &= \mathbf{E}_x \left[\int_1^2 \mathbf{P}_{X(T_i)}(\Phi_t \in A) dt \right] \\ &= \mathbf{E}_x \left[\int_1^2 \mu_{t, X(T_i)}(A) dt \right] \\ &\geq \gamma\varphi(A). \end{aligned}$$

Let $\theta := \gamma\varphi(A) \in (0, 1)$, since $\gamma \in (0, 1)$ and φ is a probability measure. Using the strong Markov property and Tonelli's theorem, it follows that for any $i \geq 1$

$$\begin{aligned} \theta &\leq \mathbf{E}_x[\eta_A^i] \\ &= \mathbf{E}_x[\eta_A^i \mathbb{1}_{\{\eta_A^i \leq \theta^2\}}] + \mathbf{E}_x[\eta_A^i \mathbb{1}_{\{\eta_A^i > \theta^2\}}] \\ &= \mathbf{E}_x[\eta_A^i \mathbb{1}_{\{\eta_A^i \leq \theta^2\}}] + \mathbf{E}_x \left[\mathbf{E}_{X(T_i)}[\eta_A^1 \mathbb{1}_{\{\eta_A^1 > \theta^2\}}] \right] \\ &\leq \theta^2 \mathbf{P}_x(\eta_A^i \leq \theta^2) + \mathbf{E}_x \left[\int_1^2 \mathbf{P}_{X(T_i)}(\Phi_t \in A, \eta_A^1 > \theta^2) dt \right] \\ &\leq \theta^2 \mathbf{P}_x(\eta_A^i \leq \theta^2) + \mathbf{E}_x \left[\mathbf{P}_{X(T_i)}(\eta_A^1 > \theta^2) \right] \\ &= \theta^2(1 - \mathbf{P}_x(\eta_A^i > \theta^2)) + \mathbf{P}_x(\eta_A^i > \theta^2) \\ &= \theta^2 + (1 - \theta^2) \mathbf{P}_x(\eta_A^i > \theta^2). \end{aligned}$$

where in the second to last equality, we have used the strong Markov property to obtain the identity $\mathbf{E}_x \left[\mathbf{P}_{X(T_i)}(\eta_A^1 > \theta^2) \right] = \mathbf{P}_x(\eta_A^i > \theta^2)$. Rearranging the above inequality, we now have

$$\mathbf{P}_x(\eta_A^i > \theta^2) \geq \frac{\theta - \theta^2}{1 - \theta^2} > 0, \quad \forall i \geq 1. \quad (3.12)$$

Moreover, the same proof shows that almost surely

$$\mathbf{P}_x(\eta_A^i > \theta^2 \mid \mathcal{F}_{T_{i-1}}) \geq \frac{\theta - \theta^2}{1 - \theta^2}$$

for all $i \geq 1$. Since the random variables $(\eta_A^j, 1 \leq j < i)$ are $\mathcal{F}_{T_{i-1}}$ -measurable, it follows that almost surely

$$\mathbf{P}_x(\eta_A^i > \theta^2 \mid \eta_A^j, 1 \leq j < i) \geq \frac{\theta - \theta^2}{1 - \theta^2}.$$

This implies that the collection of indicators $(\mathbf{1}_{[\eta_A^i > \theta^2]}, i \geq 1)$ stochastically dominates a sequence $(B_i : i \geq 1)$ of independent Bernoulli random variables success probability $\frac{\theta - \theta^2}{1 - \theta^2} > 0$. It is immediate that, almost surely, $\eta_A^i > \theta^2$ for infinitely many i , and so

$$\eta_A \geq \sum_{i \geq 1} \eta_A^i \stackrel{\text{a.s.}}{=} \infty,$$

as required. \square

4. Open questions

- It would be nice to understand the behavior of the rescaling $\sigma(d, N)$ obtained in Theorem 1.1, especially as $N \rightarrow \infty$. We have shown that $\sigma(d, 1) = \sigma(d, 2) = 1$ for all $d \geq 1$, and that in general, $\sigma(d, N)$ is related to the quantity $\Sigma(d, N) = \mathbf{E}_x[\tau_2 - \tau_1]^{-1/2} Q$, where Q is a d -by- d matrix such that $C = QQ^T$ is the covariance matrix of $\Delta_1 = X_1(\tau_2) - X_1(\tau_1)$. Thus, one way to gain information about $\sigma(d, N)$ would be to understand the relations

between the coordinates of Δ_1 . This approach seems rather difficult to implement given the abstract nature of Δ_1 . However, it seems likely that

$$\lim_{N \rightarrow \infty} \sigma(d, N) = 0, \quad (4.1)$$

for all dimensions $d \geq 1$.

- For Borel $A \subset \mathbb{R}^d$, write

$$\pi_t^N(A) := \frac{1}{N} \#\{\{X_1(t) - \bar{X}(t), \dots, X_N(t) - \bar{X}(t)\} \cap A\}$$

for the empirical measure of the BBB process viewed from its barycentre. We expect that π_t^N converges weakly as first $t \rightarrow \infty$, then $N \rightarrow \infty$, to a continuous Borel measure with compact support. It seems likely that the limiting measure is the one identified as the hydrodynamic limit of the branching particle systems studied in [5, 6], where the authors consider N -BBMs with killing of the particle furthest from the origin.

5. Appendix

Given two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ with $g \neq 0$, we write $f \stackrel{m \rightarrow \infty}{\sim} g$ if $\frac{f(m)}{g(m)} = 1 + o(1)$ as $m \rightarrow \infty$. Moreover, we define the set

$$\Lambda := \{\lambda : [0, 1] \rightarrow [0, 1] : \lambda \text{ is a strictly increasing, continuous bijection}\},$$

and subsequently the Skorohod metric on $\mathcal{D}([0, 1], \mathbb{R}^d)$ by

$$d(x, y) := \inf_{\lambda \in \Lambda} \{\max(d_\infty(\lambda, I), d_\infty(x, y \circ \lambda))\},$$

where $I \in \Lambda$ is the identity map on $[0, 1]$, and d_∞ is the supremum metric defined by

$$d_\infty(f, g) := \sup_{0 \leq t \leq 1} |f(t) - g(t)|$$

for functions f and g on $[0, 1]$. In particular, we observe that for all $x, y \in \mathcal{D}([0, 1], \mathbb{R}^d)$

$$d(x, y) \leq d_\infty(x, y) \quad (5.1)$$

by taking $\lambda = I \in \Lambda$. We can now recall the multidimensional version of Donsker's invariance principle.

Theorem 5.1. (Donsker's Theorem, Theorem 4.3.5 in [15]) *Let $(\Delta_i; i \geq 1)$ be a sequence of independent and identically distributed \mathbb{R}^d -valued random variables with mean β and d -by- d covariance matrix $C = QQ^T$, for some matrix Q . Define the partial sums $S(j) := \sum_{i=1}^j \Delta_i$. Then, we have*

$$\left(m^{-1/2}(S(\lfloor tm \rfloor) - tm\beta), 0 \leq t \leq 1\right) \xrightarrow{d} (QB(t), 0 \leq t \leq 1)$$

in the Skorohod topology on $\mathcal{D}([0, 1], \mathbb{R}^d)$ as $m \rightarrow \infty$, where $B = (B(t), 0 \leq t \leq 1)$ is a standard d -dimensional Brownian motion.

5.1. Proof of Proposition 1.4

Let $x \in \mathbb{R}^{d \times N}$ denote the initial configuration of X , i.e. $X(0) = x$. The main idea will be to approximate \bar{X} by a sum of IID random variables, for which we may apply Donsker's invariance principle. The approximation is justified by properties (1)-(5) in Proposition 1.4. Throughout the appendix, we denote by $B = (B(t), t \geq 0)$ a standard d -dimensional Brownian motion and we fix $\mathfrak{m} := \mathbf{E}_x[\tau_2 - \tau_1]$.

We first define the appropriate sum of IID random variables which automatically satisfies an invariance principle.

Lemma 5.2. *Consider the sequence of sums of increments $S(j) := \sum_{i=1}^j \Delta_i$, where $\Delta_i = X_1(\tau_{i+1}) - X_1(\tau_i)$. There exist $\beta \in \mathbb{R}^d$ and a d -by- d matrix Q such that*

$$\left(m^{-1/2} (S(\lfloor tm \rfloor) - tm\beta), 0 \leq t \leq 1 \right) \xrightarrow{d} (QB(t), 0 \leq t \leq 1)$$

in the Skorokod topology on $\mathcal{D}([0, 1], \mathbb{R}^d)$ as $m \rightarrow \infty$.

Proof. By property (2), the random variables $(\Delta_i; i \geq 1)$ are IID. Moreover, by property (4), they belong to $L^2(\mathbf{P}_x)$. Letting $C = QQ^T$ be the covariance matrix of Δ_1 and taking $\beta = \mathbf{E}_x[\Delta_1]$, the lemma then follows from Donsker's invariance principle (Theorem 5.1). \square

For each $s \geq \tau_1$, let $k[s]$ be the unique positive integer such that $\tau_{k[s]} \leq s < \tau_{k[s]+1}$ and for $s < \tau_1$ let $k[s] = 0$. Note that each $k[s]$ is a random variable. We next control the errors in probability between an appropriate rescaling of X_1 and S .

Lemma 5.3. *Fix any initial configuration $x \in \mathbb{R}^{d \times N}$. We have*

$$d_\infty \left(\frac{X_1(\cdot m)}{\sqrt{m}}, \frac{S(k[\cdot m])}{\sqrt{m}} \right) \rightarrow 0$$

in probability with respect to \mathbf{P}_x as $m \rightarrow \infty$.

Proof. Since $\tau_1 < \infty$ almost surely with respect to \mathbf{P}_x and X is right-continuous, it suffices to prove that as $m \rightarrow \infty$

$$m^{-1/2} \sup_{\tau_1 \leq t \leq m} |X_1(t) - S(k[t])| \rightarrow 0$$

in probability with respect to \mathbf{P}_x . We have

$$\begin{aligned} X_1(t) - S(k[t]) &= X_1(t) - \sum_{i=1}^{k[t]} (X_1(\tau_{i+1}) - X_1(\tau_i)) \\ &= X_1(t) - X_1(\tau_{k[t]+1}) + X_1(\tau_1) \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{\tau_1 \leq t \leq m} |X_1(t) - S(k[t])| &= \sup_{\tau_1 \leq t \leq m} |X_1(t) - X_1(\tau_{k[t]+1}) + X_1(\tau_1)| \\ &\leq \max_{1 \leq i \leq k[m]} \sup_{\tau_i \leq s \leq \tau_{i+1}} |X_1(s) - X_1(\tau_i)| + |X_1(\tau_1)| \end{aligned}$$

almost surely. Fix any $\epsilon > 0$. It follows that

$$\begin{aligned} &\mathbf{P}_x \left(\sup_{\tau_1 \leq t \leq m} m^{-1/2} |X_1(t) - S(k[t])| > \epsilon \right) \\ &\leq \mathbf{P}_x \left(\max_{1 \leq i \leq k[m]} \sup_{\tau_i \leq s \leq \tau_{i+1}} m^{-1/2} |X_1(s) - X_1(\tau_i)| > \frac{\epsilon}{2} \right) + \mathbf{P}_x \left(m^{-1/2} |X_1(\tau_1)| > \frac{\epsilon}{2} \right) \end{aligned} \quad (5.2)$$

Since $\tau_1 < \infty$ almost surely with respect to \mathbf{P}_x and X is right-continuous, then $|X_1(\tau_1)| < \infty$ almost surely with respect to \mathbf{P}_x . Therefore, the second probability in (5.2) converges to 0 as $m \rightarrow \infty$. Consider the random variables

$$Z_i := \sup_{\tau_i \leq s \leq \tau_{i+1}} |X_1(s) - X_1(\tau_i)|, \quad i = 1, \dots, k[m].$$

By property (3) of Proposition 1.4, the $(Z_i)_{i \geq 1}$ are identically distributed. Moreover, by property (4) of Proposition 1.4 and the triangle inequality, we obtain $\mathbf{E}_x[Z_1^2] < \infty$. Since $(\tau_{i+1} - \tau_i; i \geq 1)$ are IID with finite mean by Property (1) of Proposition 1.4, the strong law of large numbers implies that $\frac{\tau_n}{n} \rightarrow \mathbf{m}$, \mathbf{P}_x -a.s. as $n \rightarrow \infty$. It then easily follows that

$$k[s] \stackrel{s \rightarrow \infty}{\sim} \mathbf{m}^{-1} s, \quad \mathbf{P}_x\text{-a.s.} \quad (5.3)$$

The result therefore follows from Fact 2.12. \square

Combining Lemma 5.3 with property (5) of Proposition 1.4, we obtain the following lemma.

Lemma 5.4. *We have*

$$d_\infty \left(\frac{\bar{X}(\cdot m)}{\sqrt{m}}, \frac{S(k[\cdot m])}{\sqrt{m}} \right) \rightarrow 0$$

in probability as $m \rightarrow \infty$. In particular, by (5.1), this implies that

$$d \left(m^{-1/2} (\bar{X}(\cdot m) - k[\cdot m]\beta), m^{-1/2} (S(k[\cdot m]) - k[\cdot m]\beta) \right) \rightarrow 0$$

in probability as $m \rightarrow \infty$, where $\beta := \mathbf{E}_x[\Delta_1]$.

Proof. For every $\epsilon > 0$, the triangle inequality implies that

$$\begin{aligned} & \mathbf{P}_x \left(d_\infty \left(\frac{\bar{X}(\cdot m)}{\sqrt{m}}, \frac{S(k[\cdot m])}{\sqrt{m}} \right) > \epsilon \right) \\ & \leq \mathbf{P}_x \left(d_\infty \left(\frac{\bar{X}(\cdot m)}{\sqrt{m}}, \frac{X_1(\cdot m)}{\sqrt{m}} \right) > \frac{\epsilon}{2} \right) + \mathbf{P}_x \left(d_\infty \left(\frac{X_1(\cdot m)}{\sqrt{m}}, \frac{S(k[\cdot m])}{\sqrt{m}} \right) > \frac{\epsilon}{2} \right) \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$. \square

We will make use of the following version of Slutsky's Theorem for general metric space-valued random variables.

Theorem 5.5. (Slutsky's Theorem, Theorem 3.1, [8]) *Let (M, d) be a metric space. Suppose that (X_n, Y_n) are random elements of $M \times M$. If $X_n \xrightarrow{d} X$ and $d(X_n, Y_n) \rightarrow 0$ in probability, then $Y_n \xrightarrow{d} X$.*

We are now able to prove Proposition 1.4.

Proof of Proposition 1.4. In this proof, every convergence in distribution statement is with respect to the Skorohod topology on $\mathcal{D}([0, 1], \mathbb{R}^d)$. Lemma 5.2 and (5.3) imply that

$$\begin{aligned} & \left(m^{-1/2} (S(k[tm]) - k[tm]\beta), 0 \leq t \leq 1 \right) \\ & \stackrel{m \rightarrow \infty}{\rightsquigarrow} \left(m^{-1/2} (S(\lfloor tm\mathbf{m}^{-1} \rfloor) - tm\mathbf{m}^{-1}\beta), 0 \leq t \leq 1 \right) \\ & \xrightarrow{d} \left(\mathbf{m}^{-1/2} QB(t) : 0 \leq t \leq 1 \right) \end{aligned}$$

as $m \rightarrow \infty$. Combining this with Lemma 5.4, we can apply Slutsky's Theorem (Theorem 5.5) to obtain

$$\left(m^{-1/2} (\bar{X}(tm) - k[tm]\beta), 0 \leq t \leq 1 \right) \xrightarrow{d} \left(\mathbf{m}^{-1/2} QB(t), 0 \leq t \leq 1 \right)$$

as $m \rightarrow \infty$. Hence

$$\left(m^{-1/2} (\bar{X}(tm) - tm\alpha), 0 \leq t \leq 1 \right) \xrightarrow{d} (\Sigma B(t), 0 \leq t \leq 1)$$

as $m \rightarrow \infty$, where in view of (5.3), we have defined

$$\alpha := \beta \mathbf{m}^{-1} = \mathbf{E}_x[\Delta_1] \mathbf{m}^{-1} \quad \text{and} \quad \Sigma := \mathbf{m}^{-1/2} Q. \quad \square$$

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