Math 589 – Winter 2020 – Assignment 3

Assigned on February 14, 2020.
Due on February 24, 2020 at 2 PM (by email or in my office)

1. [How many leaves does a tree have?] For a node v of a tree t, say that v is a *leaf* of t if $c(v;t) = 0$. Fix an offspring distribution μ and write $\alpha = \sum_{i \geq 0} i\mu(i)$. Then let T be a GW_μ-distributed random tree, and for $n \geq 0$ let $L_n = |\{v \in T_n : v \text{ is a leaf of } T\}|$.

   Suppose that $\mu(0) > 0$.

   (a) Prove that as $n \to \infty$,

   $$\frac{L_n}{|T_n|} \mathbf{1}_{[M \neq 0]} \stackrel{\text{a.s.}}{\to} \mu(0) \mathbf{1}_{[M \neq 0]} ,$$

   where $M := \limsup M_n$ is the almost sure limit of the martingale $M_n = |T_n|/\alpha^n$. Interpret the ratio as zero if $|T_n| = 0$.

   Proof. If $\alpha \leq 1$ then by the fundamental theorem of branching processes either $\mu(1) = 1$ or else $M \stackrel{\text{a.s.}}{=} 0$; in both cases the claim is obvious. We hereafter restrict attention to the case $\alpha > 1$.

   Fix $k \geq 0$, and fix any tree t with $|t_n| = k$. Conditionally given that $T_n = t_n$, we have

   $$\{v \in T_n : v \text{ is a leaf of } T\} = \{v \in t_n : B_v = 0\}.$$

   The distribution of $|\{v \in t_n : B_v = 0\}|$ is $\mathrm{Binomial}(k, \mu(0))$ since the random variables B_v are iid and μ-distributed, so with X a $\mathrm{Binomial}(k, \mu(0))$ random variable,

   $$\mathbf{P}\left\{ \left| \frac{L_n}{|T_n|} - \mu(0) \right| > \epsilon \;\middle|\; T_n = t_n \right\} = \mathbf{P}\{|X - k\mu(0)| \geq \epsilon k\}$$
   $$\leq \frac{\mathrm{Var}(X)}{(\epsilon k)^2} = \frac{k\mu(0)(1-\mu(0))}{(\epsilon k)^2} \leq \frac{\mu(0)}{\epsilon^2 k} .$$

   Since this holds for any tree t with $|t_n| = k$, by the law of total probability it follows that

   $$\mathbf{P}\left\{ \left| \frac{L_n}{|T_n|} - \mu(0) \right| > \epsilon \;\middle|\; |T_n| = k \right\} \leq \frac{\mu(0)}{\epsilon k} . \qquad (1)$$ chebyshev_b

   Now fix $\ell > 0$, and let

   $$E_\ell = \{M > 1/\ell\} \cap \{\limsup_{n \to \infty} |L_n/|T_n| - \mu(0)| > 1/\ell\}.$$

   For $\omega \in \Omega$, if

   $$\frac{L_n(\omega)}{|T_n(\omega)|} \mathbf{1}_{[M(\omega) \neq 0]} \not\to \mu(0) \mathbf{1}_{[M(\omega) \neq 0]}$$

then there is $\ell \in \mathbb{N}$ such that E_ℓ occurs (i.e. such that $\omega \in E_\ell$). By subadditivity of probabilities, it follows that

$$\mathbf{P}\left\{\frac{L_n}{|T_n|}\mathbf{1}_{[M\neq 0]} \not\to \mu(0)\mathbf{1}_{[M\neq 0]}\right\} \leq \sum_{\ell\geq 1} E_\ell.$$

Finally, since $M_n \stackrel{\text{a.s.}}{\to} M$,

$$\begin{aligned}\mathbf{P}\{E_\ell\} &= \mathbf{P}\left\{\liminf_{n\to\infty} M_n > \frac{1}{\ell}, \limsup_{n\to\infty}\left|\frac{L_n}{|T_n|} - \mu(0)\right| > 1/\ell\right\} \\ &\leq \mathbf{P}\left\{M_n > \frac{1}{\ell} \text{ and } \left|\frac{L_n}{|T_n|} - \mu(0)\right| > \frac{1}{\ell} \text{ infinitely often}\right\}.\end{aligned}$$

To complete the proof, we use the first Borel-Cantelli lemma to show that the last probability is zero for all ℓ. To apply that lemma, we simply need to show that

$$\sum_{n\geq 1}\mathbf{P}\left\{M_n > \frac{1}{\ell} \text{ and } \left|\frac{L_n}{|T_n|} - \mu(0)\right| > \frac{1}{\ell}\right\} < \infty.$$

But $M_n = |T_n|/\alpha^n$, so by (1),

$$\begin{aligned}\mathbf{P}\left\{M_n > \frac{1}{\ell} \text{ and } \left|\frac{L_n}{|T_n|} - \mu(0)\right| > \frac{1}{\ell}\right\} &= \mathbf{P}\left\{|T_n| > \frac{\alpha^n}{\ell}, \left|\frac{L_n}{|T_n|} - \mu(0)\right| > \frac{1}{\ell}\right\} \\ &\leq \sup_{k>\alpha^n/\ell} \mathbf{P}\left\{\left|\frac{L_n}{|T_n|} - \mu(0)\right| > \frac{1}{\ell} \;\middle|\; |T_n| = k\right\} \\ &\leq \sup_{k>\alpha^n/\ell} \frac{\ell\mu(0)}{k} < \frac{\ell^2\mu(0)}{\alpha^n}.\end{aligned}$$

Since $\alpha > 1$, this upper bound is summable, as required.

(b) Prove that as $n \to \infty$,

$$\frac{L_n}{\alpha^n}\mathbf{1}_{[M\neq 0]} \stackrel{\text{a.s.}}{\to} M\mu(0)\,.$$

where $M := \limsup M_n$ is the almost sure limit of the martingale $M_n = |T_n|/\alpha^n$.
Proof. We may rewrite the conclusion of part (a) as

$$\frac{L_n}{\alpha^n}\cdot\frac{1}{M_n}\mathbf{1}_{[M\neq 0]} \stackrel{\text{a.s.}}{\to} \mu(0)\mathbf{1}_{[M\neq 0]}\,. \tag{2}$$ equa

We now apply the following straightforward fact from analysis: for real sequences $(a_n, n \geq 1)$ and $(b_n, n \geq 1)$, if $a_n/b_n \to c$ and $b_n \to b \in (0,\infty)$, then $a_n \to cb$. This implies that for all ω such that $M_n(\omega) \to M(\omega) > 0$ and such that the convergence in (2) holds, we have

$$\frac{L_n(\omega)}{\alpha^n} \to M(\omega)\mu(0).$$

When $M(\omega) = 0$, on the other hand, there is nothing to prove. Since $\mathbf{P}\{M_n \to M\} = 1$, the result follows.

2. [Moment generating function examples]

(a) Let E be an Exponential(α) random variable, so E has density $\alpha e^{-\alpha x}$ on $(0, \infty)$. Show that

$$G_E(s) = \sum_{k=0}^{\infty} \frac{s^k}{\alpha^k},$$

for $s \in (-\infty, \alpha)$.

(b) Let X be Gamma(α, u)-distributed, so X has density

$$\frac{\alpha^u}{\Gamma(u)} x^{u-1} e^{-\alpha x}$$

on $(0, \infty)$. Show that $G_X(s) = (1 - s/\alpha)^{-u}$ for $s \in (-\infty, \alpha)$, and deduce that $\mathbf{E}\left[X^k\right] = u(u+1) \cdot \ldots \cdot (u+k-1)/\alpha^k$. (Notice the relation with part (a).)

3. [Characteristic function facts]

In class and in the notes you have seen that for any a real random variable X, for all $n \geq 0$, if $\mathbf{E}[|X|^n] < \infty$ then

$$\left|\varphi_X(t) - \sum_{k=0}^{n} \frac{(it)^k}{k!} \mathbf{E}\left[X^k\right]\right| \leq \mathbf{E} \min\left(\frac{|tX|^{n+1}}{(n+1)!}, \frac{2|tX|^n}{n!}\right). \qquad (3)$$

eq:error_bo

Use this bound to prove the following.

(a) [The characteristic function of simple random walk.] Let $(X_i, i \geq 1)$ be independent fair coin tosses, i.e., $\mathbf{P}\{X_i = 1\} = \mathbf{P}\{X_i = -1\} = 1/2$, and let $S_n = n^{-1/2} \sum_{i=1}^{n} X_i$. Prove that for all $t \in \mathbb{R}$, $\varphi_{S_n}(t) \to e^{-t^2/2}$.

(b) [Finite moment generating function gives Taylor expansion of characteristic function. Let X be any random variable. Suppose that the moment generating function G_X is finite: $G_X(s) \in \mathbb{R}$ for all $s \in \mathbb{R}$. Prove that

$$\varphi_X(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \mathbf{E}\left[X^k\right]$$

for all $t \in \mathbb{R}$. (Hint: use the assumption to show the error term in (3) tends to zero as $n \to \infty$.)

4. Characteristic functions and densities

Let X be a real random variable. Show that if X has a density then $|\varphi_X(t)| \to 0$ as $t \to \infty$. Also show that if $\int_{\mathbb{R}} |\varphi_X(t)| < \infty$ then the function defined by

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \varphi_X(t) \mathrm{d}t$$

is a continuous density for X.

The next two questions are about periodicity and characteristic functions.

5. Let X be a real random variable with characteristic function $\varphi = \varphi_X$, and fix $r > 0$ Prove that the following statements are equivalent.

(i) $\varphi(r) = 1$

(ii) $\varphi(t+r) = \varphi(t)$ for all $t \in \mathbb{R}$.

(iii) $\mathbf{P}\{rX/(2\pi) \in \mathbb{Z}\} = 1$.

Hint: what does (i) imply about the probability $\mathbf{P}\{\cos(rX) < 1\}$?

Solution.

First suppose that $\varphi(r) = 1$. Then

$$\mathbf{E}[\cos(rX)] + i\mathbf{E}[\sin(rX)] = 1,$$

which means $\mathbf{E}[\cos(rX)] = 1$ and $\mathbf{E}[\sin(rX)] = 0$. But deterministically $\cos(rX) \leq 1$. If a random variable Z is deterministically at most one and $\mathbf{E}[Z] = 1$ then $Z \overset{\text{a.s.}}{=} 1$; so

$$\mathbf{P}\{\cos(rX) = 1\} = 1.$$

In other words, $\mathbf{P}\{rX/(2\pi) \in \mathbb{Z}\} = 1$, so (i) implies (iii).

Next, if $\mathbf{P}\{rX/(2\pi) \in \mathbb{Z}\} = 1$ then for all $t \in \mathbb{R}$,

$$\varphi(t+r) = \mathbf{E}\left[e^{i(t+r)X}\right] = \mathbf{E}\left[e^{itX}e^{irX}\right] = \mathbf{E}\left[e^{itX}\right] = \varphi(t)\,,$$

the third equality holding since $e^{irX} = e^{2\pi i \cdot rX/(2\pi)} \overset{\text{a.s.}}{=} 1$ by the assumption that $rX/(2\pi) \in \mathbb{Z}$ almost surely. This shows that (iii) implies (ii).

Finally, if $\varphi(r) \neq 1$ then taking $t = -r$ we have

$$\varphi(t+r) = \varphi(0) = 1 \neq \varphi(r).$$

This shows that if (i) fails then (ii) does as well; so (ii) implies (i).

6. Let X be a real random variable with characteristic function $\varphi = \varphi_X$. Prove that one of the following three conditions must occur.

   (i) $|\varphi(t)| = 1$ for all $t \in \mathbb{R}$.

   (ii) $|\varphi(t)| < 1$ for $t \neq 0$.

   (iii) There is $r > 0$ such that $|\varphi(t)| < 1$ for $0 < t < r$, and $\varphi(t) = \varphi(t+r)$ for all $t \in \mathbb{R}$.

Solution.

The problem was false as originally stated. If $X = -1$ with probability $1/2$ and $X = 1$ with probability $1/2$ then $\varphi(t) = \cos(t)$. We then have $\varphi(\pi) = -1$ and $|\varphi(t)| < 1$ for $0 < t < \pi$, but $\varphi(t+\pi) = -\varphi(t)$ for $t \in \mathbb{R}$.

Item (iii) should say "there is $r > 0$ such that $|\varphi(t)| < 1$ for $0 < t < r$, and $\varphi(t)\varphi(r) = \varphi(t+r)$ for all $t \in \mathbb{R}$."

Suppose (i) and (ii) fail; we will show that (iii) holds.

Since (ii) fails, we may choose $r \neq 0$ such that $|\varphi(r)| = 1$. Since $\varphi(-r) = \overline{\varphi(r)}$, we also have $|\varphi(-r)| < 1$, so we can in fact choose $r > 0$.

Since $|\varphi(r)| = 1$ there is $c \in [0, 1)$ such that

$$\varphi(r) = \mathbf{E}\left[e^{irX}\right] = e^{2\pi i c}.$$

Let $Y = X - 2\pi c/r$. Then for any $s \in \mathbb{R}$,

$$\varphi_Y(s) = \mathbf{E}\left[e^{isY}\right] = \mathbf{E}\left[e^{is(X-2\pi c/r)}\right] = \varphi_X(s) \cdot e^{-2\pi ics/r}.$$

In particular,

$$\varphi_Y(r) = \varphi_X(r) \cdot e^{-2\pi ic} = 1.$$

The conclusion of question 5 then implies that $\varphi_Y(t+r) = \varphi_Y(t)$ for all $t \in \mathbb{R}$, and that $\mathbf{P}\{rY/(2\pi) \in \mathbb{Z}\} = 1$.

Since $\varphi_Y(t) = \varphi_X(t) \cdot e^{-t\pi ict/r}$ and $\varphi_Y(t+r) = \varphi_X(t+r) \cdot e^{-2\pi ic(t+r)/r}$, this means that

$$\varphi_X(t+r) = \varphi_X(t) \cdot e^{2\pi ic},$$

for all $t \in \mathbb{R}$.

Since $rY/(2\pi) = rX/(2\pi) - c$, we may also conclude that in this case

$$\mathbf{P}\left\{X \in \left\{\frac{2\pi(z+c)}{r} : z \in \mathbb{Z}\right\}\right\} = 1\,. \tag{4}$$

eq:char_fn_

Now consider any $r' > 0$ such that $|\varphi(r')| = 1$, and write $\varphi(r') = e^{2\pi ic'}$ for $c' \in [0,1)$. Then the above argument shows that

$$\mathbf{P}\left\{X \in \left\{\frac{2\pi(z+c')}{r'} : z \in \mathbb{Z}\right\}\right\} = 1\,. \tag{5}$$

eq:char_fn_

For $k, \ell \in \mathbb{Z}$, if

$$\frac{2\pi(k+c)}{r} = \frac{2\pi(\ell+c')}{r'}$$

then

$$\ell = (k+c)\frac{r'}{r} - c'.$$

But if r'/r is irrational then there is at most one integer k such that $(k+c)r'/r - c'$ is integer. (In other words, the lattices in (4) and in (5) have at most one point in common.) So if r'/r is irrational then (4) and (5) together imply that X is almost surely constant. But in this case $|\varphi(t)| = 1$ for all t, contradicting (i).

We may thus assume that $|\varphi(t)| = 1$ only at rational multiples of r. Now let $r^* = \inf\{s > 0 : |\varphi(s)| = 1\}$. If $r^* = 0$ then this means we may find $r > 0$ arbitrarily small such that $|\varphi(r)| = 1$. But we saw that for such r, we have $|\varphi(kr)| = 1$ for all integer k. Thus if $r^* = 0$ we may find a dense subset S of $\mathbb{R}$ such that $|\varphi(s)| = 1$ for all $s \in S$. The continuity of φ then implies that $|\varphi(r)| = 1$ for all $r \in \mathbb{R}$, again contradicting (i).

We may now assume that $r^* > 0$. The continuity of φ and the definition of r^* together implies that $|\varphi(r^*)| = 1$ and $|\varphi(s)| < 1$ for $s \in (0, r)$. So we may write

$$\varphi(r^*) = e^{2\pi ic^*}$$

for some $c^* \in [0,1)$, and as above it follows that $\varphi(t + r^*) = e^{2\pi ic^*}\varphi(t) = \varphi(r^*)\varphi(t)$ for all $t \in \mathbb{R}$.