Math 589 – Winter 2020 – Assignment 3

Assigned on February 14, 2020.
Due on February 24, 2020 at 2 PM (by email or in my office)

1. [How many leaves does a tree have?] For a node v of a tree t, say that v is a *leaf* of t if $c(v;t) = 0$. Fix an offspring distribution μ and write $\alpha = \sum_{i\geq 0} i\mu(i)$. Then let T be a GW_μ-distributed random tree, and for $n \geq 0$ let $L_n = |\{v \in T_n : v \text{ is a leaf of } T\}|$. Suppose that $\mu(0) > 0$.

   (a) Prove that as $n \to \infty$,
   $$\frac{L_n}{|T_n|}\mathbf{1}_{[M\neq 0]} \stackrel{\text{a.s.}}{\to} \mu(0)\mathbf{1}_{[M\neq 0]}\,,$$
   where $M := \limsup M_n$ is the almost sure limit of the martingale $M_n = |T_n|/\alpha^n$. Interpret the ratio as zero if $|T_n| = 0$.

   (b) Prove that as $n \to \infty$,
   $$\frac{L_n}{\alpha^n}\mathbf{1}_{[M\neq 0]} \stackrel{\text{a.s.}}{\to} M\mu(0)\,.$$
   where $M := \limsup M_n$ is the almost sure limit of the martingale $M_n = |T_n|/\alpha^n$.

2. [Moment generating function examples]

   (a) Let E be an Exponential(α) random variable; this means E is non-negative and has density $\alpha e^{-\alpha x}$ on $(0, \infty)$. Show that
   $$G_E(s) = \sum_{k=0}^{\infty} \frac{s^k}{\alpha^k}\,,$$
   for $s \in (-\infty, \alpha)$.

   (b) Let X be Gamma(α, u)-distributed; this means X is non-negative and has density
   $$\frac{\alpha^u}{\Gamma(u)}x^{u-1}e^{-\alpha x}$$
   on $(0, \infty)$. Show that $G_X(s) = (1 - s/\alpha)^{-u}$ for $s \in (-\infty, \alpha)$, and deduce that $\mathbf{E}\left[X^k\right] = u(u + 1) \cdot \ldots \cdot (u + k - 1)/\alpha^k$. (Notice the relation with part (a).)

3. [Characteristic function facts]

   In class and in the notes you have seen that for any a real random variable X, for all $n \geq 0$, if $\mathbf{E}\left[|X|^n\right] < \infty$ then
   $$\left|\varphi_X(t) - \sum_{k=0}^{n} \frac{(it)^k}{k!}\mathbf{E}\left[X^k\right]\right| \leq \mathbf{E}\min\left(\frac{|tX|^{n+1}}{(n+1)!}, \frac{2|tX|^n}{n!}\right)\,. \qquad (1)$$
   eq:error_bo

   Use this bound to prove the following.

(a) [The characteristic function of simple random walk.] Let $(X_i, i \geq 1)$ be independent fair coin tosses, i.e., $\mathbf{P}\{X_i = 1\} = \mathbf{P}\{X_i = -1\} = 1/2$, and let $S_n = n^{-1/2} \sum_{i=1}^{n} X_i$. Prove that for all $t \in \mathbb{R}$, $\varphi_{S_n}(t) \to e^{-t^2/2}$.

(b) [Finite moment generating function gives Taylor expansion of characteristic function. Let X be any random variable. Suppose that the moment generating function G_X is finite: $G_X(s) \in \mathbb{R}$ for all $s \in \mathbb{R}$. Prove that

$$\varphi_X(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \mathbf{E}\left[X^k\right]$$

for all $t \in \mathbb{R}$. (Hint: use the assumption to show the error term in (1) tends to zero as $n \to \infty$.)

4. Characteristic functions and densities

Let X be a real random variable. Show that if X has a density with respect to Lebesgue measure then $|\varphi_X(t)| \to 0$ as $t \to \infty$. Also show that if $\int_{\mathbb{R}} |\varphi_X(t)| < \infty$ then the function defined by

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \varphi_X(t) \mathrm{d}t$$

is a continuous density for X.

The next two questions are about periodicity and characteristic functions.

5. Let X be a real random variable with characteristic function $\varphi = \varphi_X$, and fix $r > 0$ Prove that the following statements are equivalent.

   (i) $\varphi(r) = 1$

   (ii) $\varphi(t + r) = \varphi(t)$ for all $t \in \mathbb{R}$.

   (iii) $\mathbf{P}\{rX/(2\pi) \in \mathbb{Z}\} = 1$.

   Hint: what does (i) imply about the probability $\mathbf{P}\{\cos(rX) < 1\}$?

6. Let X be a real random variable with characteristic function $\varphi = \varphi_X$. Prove that one of the following three conditions must occur.

   (i) $|\varphi(t)| = 1$ for all $t \in \mathbb{R}$.

   (ii) $|\varphi(t)| < 1$ for $t \neq 0$.

   (iii) There is $r > 0$ such that $|\varphi(t)| < 1$ for $0 < t < r$, and $\varphi(t)\varphi(r) = \varphi(t + r)$ for all $t \in \mathbb{R}$.