Math 589 – Winter 2020 – Assignment 2

Assigned on January 31, 2020.
Due on February 9, 2020 at 2 PM (by email or in my office)

1. Let $(Z_n, n \geq 0)$ be the generation sizes in a Galton-Watson process with offspring distribution μ. Let B be μ-distributed and write $\alpha = \mathbf{E}B$ and $\sigma^2 = \mathbf{Var}\{B\}$. We suppose in this question that $\sigma^2 \in (0, \infty)$ and that $\alpha > 1$. Also, write $M_n = Z_n/(\mathbf{E}B)^n$ and let M be the a.s. martingale limit of M_n.

   (a) Prove that for every $n \geq 0$,

   $$\mathbf{E}\left\{ Z_{n+1}^2 \;\middle|\; \mathcal{F}_n \right\} = (\mathbf{E}B)^2 Z_n^2 + \sigma^2 Z_n.$$

   Proof. We may represent Z_{n+1} as $Z_{n+1} = \sum_{i=1}^{Z_n} B_{n+1,i}$ where $(B_{n,m}, n, m \geq 1)$ are iid μ-distributed random variables. Then by conditional linearity of expectation,

   $$\begin{aligned}\mathbf{E}\left\{ Z_{n+1}^2 \;\middle|\; Z_n = k \right\} &= \mathbf{E}\left\{ \sum_{i,j=1}^{k} B_{n+1,i}B_{n+1,j} \;\middle|\; Z_n = k \right\} \\ &= \sum_{i,j=1}^{k} \mathbf{E}\left\{ B_{n+1,i}B_{n+1,j} \;\middle|\; Z_n = k \right\}.\end{aligned}$$

   But $B_{n+1,i}$ and $B_{n+1,j}$ are independent of Z_n, so

   $$\mathbf{E}\left\{ B_{n+1,i}B_{n+1,j} \;\middle|\; Z_n = k \right\} = \mathbf{E}\left[B_{n+1,i}B_{n+1,j} \right] = \begin{cases} (\mathbf{E}B)^2 & \text{if } i \neq j \\ \mathbf{E}\left[B^2\right] & \text{if } i = j. \end{cases}$$

   Since $\mathbf{E}\left[B^2\right] = \sigma^2 + (\mathbf{E}B)^2$, it follows that

   $$\mathbf{E}\left\{ Z_{n+1}^2 \;\middle|\; Z_n = k \right\} = k^2(\mathbf{E}B)^2 + k\sigma^2.$$

   The result follows since

   $$\mathbf{E}\left\{ Z_{n+1}^2 \;\middle|\; Z_n \right\} \overset{\text{a.s.}}{=} \sum_{k \geq 1} \mathbf{E}\left\{ Z_{n+1}^2 \;\middle|\; Z_n = k \right\} \mathbf{1}_{[Z_n = k]},$$

   a formula that is easily verified by the definition of conditional expectation.

   (b) Prove that for every $n \geq 0$,

   $$\mathbf{E}\left[Z_n^2\right] = \alpha^{2n} + \frac{\sigma^2(\alpha^n - \alpha^{2n})}{\alpha(1-\alpha)}$$

   Proof. The formula is obvious for $n = 0$. Fix $n \geq 0$ for which the formula holds. Then by part (a) and the tower law,

   $$\begin{aligned}\mathbf{E}\left[Z_{n+1}^2\right] &= \mathbf{E}\left[\mathbf{E}\left\{ Z_{n+1}^2 \;\middle|\; Z_n \right\}\right] \\ &= \mathbf{E}\left[(\mathbf{E}B)^2 Z_n^2 + \sigma^2 Z_n\right]\end{aligned}$$

Since $\mathbf{E}Z_n = \alpha^n$, by the assumption that the formula holds for this value of n we have

$$\mathbf{E}\left[Z_{n+1}^2\right] = \alpha^2\left(\alpha^{2n} + \frac{\sigma^2(\alpha^n - \alpha^{2n})}{\alpha(1-\alpha)}\right) + \sigma^2\alpha^n = \alpha^{2(n+1)} + \frac{\sigma^2(\alpha^{n+1} - \alpha^{2n+1})}{\alpha(1-\alpha)},$$

the last identity holding by simple arithmetic. The result follows by induction.

(c) Prove that $M_n \to M$ in L_2 and that $\mathbf{Var}\{M\} = \frac{\sigma^2}{\alpha(\alpha-1)}$. Proof. First, note that for all $n \in \mathbb{N}$,

$$\begin{aligned}\mathbf{E}\left[M_{n+1}^2\right] &= \mathbf{E}\left[(M_n + (M_{n+1} - M_n))^2\right] \\ &= \mathbf{E}\left[M_n^2\right] + 2\mathbf{E}\left[M_n(M_{n+1} - M_n\right] + \mathbf{E}\left[(M_{n+1} - M_n)^2\right] ;\end{aligned}$$

applying linearity of expectation is justified: since both M_n and $(M_{n+1} - M_n)$ are in L_2, all the above terms are in L_1. By the tower law,

$$\mathbf{E}\left[M_n(M_{n+1} - M_n)\right] = \mathbf{E}\left[\mathbf{E}\left[M_n(M_{n+1} - M_n) \mid \mathcal{F}_n\right]\right]$$

Since M_n is $\mathcal{F}_n$-measurable and $\mathbf{E}\{M_{n+1} \mid \mathcal{F}_n\} = M_n$, we have

$$\mathbf{E}\left[M_n(M_{n+1} - M_n) \mid \mathcal{F}_n\right] = M_n\mathbf{E}\left[M_{n+1} - M_n \mid \mathcal{F}_n\right] = 0\,,$$

and combining the three preceding identities gives that

$$\mathbf{E}\left[M_{n+1}^2\right] = \mathbf{E}\left[M_n^2\right] + \mathbf{E}\left[(M_{n+1} - M_n)^2\right] .$$

It follows by a telescoping sum that for all $n, k \in \mathbb{N}$,

$$\mathbf{E}\left[M_{n+k}^2\right] = \mathbf{E}\left[M_n^2\right] + \sum_{i=0}^{k-1}\mathbf{E}\left[(M_{n+i+1} - M_{n+i})^2\right]$$

But note also that

$$\mathbf{E}\left[(M_{n+k} - M_n)^2\right] = \mathbf{E}\left[M_{n+k}^2\right] - 2\mathbf{E}\left[M_{n+k}M_n\right] + \mathbf{E}\left[M_n^2\right] = \mathbf{E}\left[M_{n+k}^2\right] - \mathbf{E}\left[M_n^2\right] ;$$

the last equality holds since $\mathbf{E}\left[M_{n+k}M_n\right] = \mathbf{E}\left[M_n^2\right] + \mathbf{E}\left[M_n(M_{n+k} - M_n)\right]$, and the martingale property implies that $\mathbf{E}\left[M_n(M_{n+k} - M_n)\right] = 0$. Since $M_n^2 = Z_n/\alpha^{2n}$, using the formula for $\mathbf{E}\left[Z_n^2\right]$ from part (b) of the question we thus have

$$\begin{aligned}\mathbf{E}\left[(M_{n+k} - M_n)^2\right] &= \mathbf{E}\left[(M_{n+k})^2\right] - \mathbf{E}\left[M_n^2\right] \\ &= \sum_{i=0}^{k-1}\mathbf{E}\left[(M_{n+i+1} - M_{n+i})^2\right] \\ &= \sum_{i=0}^{k-1}\frac{\sigma^2}{\alpha^{n+i+2}}\end{aligned}$$

Since $0 \le (M_{n+k} - M_n)^2 \stackrel{\text{a.s.}}{\to} (M - M_n)^2$ as $k \to \infty$, by Fatou's lemma it follows that

$$\begin{aligned}\mathbf{E}\left[(M - M_n)^2\right] &\le \liminf_{n\to\infty}\mathbf{E}\left[(M_{n+k} - M_n)^2\right] \\ &= \sum_{i\ge 0}\mathbf{E}\left[(M_{n+i+1} - M_{n+i})^2\right] \\ &= \sum_{i\ge 0}\frac{\sigma^2}{\alpha^{n+i+2}} .\end{aligned}$$

Since $\alpha > 1$, the last sum tends to zero as $n \to \infty$; this implies that $M \in L_2$ and that $M_n \to M$ in L_2.

2. [How many leaves does a tree have?]

   This question was postponed to assignment 3.

3. [Conditioning a Galton-Watson tree to be finite.] Let T be GW_μ-distributed as in the previous question, and write $q = \mathbf{P}\{|T| < \infty\}$. Assume μ is such that $q > 0$, and define a new measure μ_{fin} on $\mathbb{N}_{\geq 0}$ by $\mu_{\text{fin}}(i) = \mu(i) \cdot q^{i-1}$.

   (a) Prove that μ_{fin} is a probability measure.

   (b) Prove that if $\sum_{i \geq 0} i\mu(i) > 1$ then $\sum_{i \geq 0} i\mu_{\text{fin}}(i) < 1$.

   (c) Let T' be $\text{GW}_{\mu_{\text{fin}}}$-distributed. Prove that the conditional distribution of T, given that extinction occurs, is that of T'. In other words, prove that for any tree t and any $n \geq 1$,

   $$\mathbf{P}\left\{T'_{\leq n} = t_{\leq n}\right\} = \mathbf{P}\left\{T_{\leq n} = t_{\leq n} \mid |T| < \infty\right\} .$$

   I have not yet had time to write a solution for this one.

4. [The effect of size-biased sampling.]
   A size-biased sample from $x_1, \ldots, x_n$ is a random variable I with $\mathbf{P}\{I = i\} \propto x_i$. This question is about the distributional properties of a size-biased sample from a collection of iid random variables.
   Let $X_1, \ldots, X_n$ be iid positive random variables (where $n \geq 3$), and let S_n be their sum. Let J be a size-biased sample from $X_1, \ldots, X_n$, so

   $$\mathbf{P}\{J = j \mid X_1, \ldots, X_n\} = X_j / S_n.$$

   This determines the joint distribution of J and $X_1, \ldots, X_n$.

   Next, for $1 \leq i \leq n-1$, let

   $$X_i^* = \begin{cases} X_i & \text{if } i < J \\ X_{i+1} & \text{if } i \geq J, \end{cases}$$

   so that $(X_1^*, \ldots, X_{n-1}^*) = (X_1, \ldots, X_{J-1}, X_{J+1}, \ldots, X_n)$ is "$(X_1, \ldots, X_n)$ with a size-biased sample removed." Let $S_{n-1}^* = X_1^* + \ldots + X_{n-1}^* = S_n - X_J$.

   (a) Prove that X_J is conditionally independent of $(X_1^*, \ldots, X_{n-1}^*)$ given S_{n-1}^*, in that for any bounded Borel functions $f : \mathbb{R}^{n-1} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$,

   $$\mathbf{E}\left[f(X_1^*, \ldots, X_{n-1}^*)g(X_J) \mid S_{n-1}^*\right] = \mathbf{E}\left[f(X_1^*, \ldots, X_{n-1}^*) \mid S_{n-1}^*\right] \mathbf{E}\left[g(X_J) \mid S_{n-1}^*\right] .$$

   (b) Prove that the conditional distribution of $(X_1^*, \ldots, X_{n-1}^*)$ given S_{n-1}^* is equivalent to the conditional distribution of $(X_1, \ldots, X_{n-1})$ given $X_1 + \ldots + X_{n-1}$, in that for any bounded Borel function $f : \mathbb{R}^{n-1} \to \mathbb{R}$,

   $$\mathbf{E}\left[f(X_1^*, \ldots, X_{n-1}^*) \mid S_{n-1}^*\right] \stackrel{\text{d}}{=} \mathbf{E}\left[f(X_1, \ldots, X_{n-1}) \mid S_{n-1}\right].$$

   (Hint: Use the definition of conditional expectation.)

   Proof.

Let f be a bounded Borel function defined on $\mathbb{R}^{n-1}$ and g, h be two bounded real valued Borel functions, then:

$$E(f(X^*_{(n-1)})g(X_J)h(S^*_{n-1})) = E(E(f(X^*_{(n-1)}))g(X_J)\,|S^*_{n-1})h(S^*_{n-1}))\,. \qquad (2.4.a)$$

On the other hand, put

$$\begin{aligned} X^{(j)}_{(n-1)} &= (X_1, X_2, \cdots, X_{j-1}, X_{j+1}, \cdots, X_n) \\ S^{(j)}_{n-1} &= \sum_{1\le i\le n,\, i\ne j} X_i\,, \end{aligned}$$

then from the hypotheses:

$$\begin{aligned} E(f(X^*_{(n-1)})g(X_J)h(S^*_{n-1})) &= \sum_{j=1}^n E(f(X^{(j)}_{(n-1)})g(X_j)h(S^{(j)}_{n-1})1_{\{J=j\}}) \\ &= \sum_{j=1}^n E\left(f(X^{(j)}_{(n-1)})g(X_j)h(S^{(j)}_{n-1})\frac{X_j}{S^{(j)}_{n-1}+X_j}\right). \end{aligned}$$

Since $X^{(j)}_{(n-1)}$ and X_j are independent, then:

$$E(f(X^*_{(n-1)})g(X_J)h(S^*_{n-1})) = \sum_{j=1}^n E\left(E(f(X^{(j)}_{(n-1)})\,|\,S^{(j)}_{n-1})g(X_j)h(S^{(j)}_{n-1})\frac{X_j}{S^{(j)}_{n-1}+X_j}\right).$$

Put $E(f(X^{(j)}_{(n-1)})\,|\,S^{(j)}_{n-1}) = k(S^{(j)}_{n-1})$, then from above,

$$E(f(X^*_{(n-1)})g(X_J)h(S^*_{n-1})) = E(k(S^*_{(n-1)})g(X_J)h(S^*_{n-1}))\,.$$

This implies:

$$\begin{aligned} E(f(X^*_{(n-1)})g(X_J)h(S^*_{n-1})) &= E(k(S^*_{(n-1)})E(g(X_J)\,|\,S^*_{n-1})h(S^*_{n-1})) \\ &= E(E(f(X^*_{(n-1)}\,|\,S^*_{(n-1)})E(g(X_J)\,|\,S^*_{n-1})h(S^*_{n-1})) \end{aligned} \qquad (2.4.b)$$

Comparing (2.4.a) and (2.4.b), we have:

$$E(f(X^*_{(n-1)})g(X_J)\,|\,S^*_{n-1}) = E(f(X^*_{(n-1)})\,|\,S^*_{n-1})E(g(X_J)\,|\,S^*_{n-1})\,,$$

thus, given S^*_{n-1}, $X^*_{(n-1)}$ and X_J are independent. Now, putting $E(f(X^*_{(n-1)})\,|\,S^*_{n-1}) = k_1(S^*_{n-1})$ and $E(f(X_{(n-1)})\,|\,S_{n-1}) = k_2(S_{n-1})$ then we shall show that $k_1(S^*_{n-1}) = k_2(S_{n-1})$ a.s. By the same arguments as above, we have:

$$E(f(X^*_{(n-1)})g(S^*_{n-1})) = \sum_{j=1}^n E\left(E(f(X^{(j)}_{(n-1)}\,|\,S^{(j)}_{n-1})g(S^{(j)}_{n-1})\frac{X_j}{S_n}\right).$$

Since the law of $X^{(j)}_{(n-1)}$ is the same as the law of $X_{(n-1)}$,

$$\begin{aligned} E(f(X^*_{(n-1)})g(S^*_{n-1})) &= \sum_{j=1}^n E\left(k_2(S^{(j)}_{n-1})g(S^{(j)}_{n-1})\frac{X_j}{S_n}\right) \\ &= E(k_2(S^*_{n-1})g(S^*_{n-1}))\,. \end{aligned}$$

But also, by definition, $E(f(X^*_{(n-1)})g(S^*_{n-1})) = E(k_1(S^*_{n-1})g(S^*_{n-1}))$, which proves the result.