Math 589 – Winter 2020 – Assignment 2

Assigned on January 31, 2020.
Due on February 9, 2020 at 2 PM (by email or in my office)

1. Exercise 13.2 from the course notes.

2. [How many leaves does a tree have?] For a node v of a tree t, say that v is a *leaf* of t if $c(v;t) = 0$. Fix an offspring distribution μ and write $\alpha = \sum_{i\geq 0} i\mu(i)$. Then let T be a GW_μ-distributed random tree, and for $n \geq 0$ let $L_n = |\{v \in T_n : v \text{ is a leaf of } T\}|$. Suppose that $\mu(0) > 0$.

   (a) Prove that as $n \to \infty$,
   $$\frac{L_n}{|T_n|} \stackrel{\text{a.s.}}{\to} \mu(0)\mathbf{1}_{[M\neq 0]}\,,$$
   where $M := \limsup M_n$ is the almost sure limit of the martingale $M_n = |T_n|/\alpha^n$. Interpret the ratio as zero if $|T_n| = 0$.
   Hint. First bound
   $$\mathbf{P}\left\{\left|\frac{L_n}{|T_n|} - \mu(0)\right| > \epsilon \;\middle|\; |T_n| = k\right\},$$
   then use information about the growth of $|T_n|$ to conclude.

   (b) Prove that as $n \to \infty$,
   $$\frac{L_n}{\alpha^n} \stackrel{\text{a.s.}}{\to} M\mu(0)\,.$$
   where $M := \limsup M_n$ is the almost sure limit of the martingale $M_n = |T_n|/\alpha^n$.

3. [Conditioning a Galton-Watson tree to be finite.] Let T be GW_μ-distributed as in the previous question, and write $q = \mathbf{P}\{|T| < \infty\}$. Assume μ is such that $q > 0$, and define a new measure μ_{fin} on $\mathbb{N}_{\geq 0}$ by $\mu_{\text{fin}}(i) = \mu(i)\cdot q^{i-1}$.

   (a) Prove that μ_{fin} is a probability measure.

   (b) Prove that if $\sum_{i\geq 0} i\mu(i) > 1$ then $\sum_{i\geq 0} i\mu_{\text{fin}}(i) < 1$.

   (c) Let T' be $\mathrm{GW}_{\mu_{\text{fin}}}$-distributed. Prove that the conditional distribution of T, given that extinction occurs, is that of T'. In other words, prove that for any tree t and any $n \geq 1$,
   $$\mathbf{P}\left\{T'_{\leq n} = t_{\leq n}\right\} = \mathbf{P}\left\{T_{\leq n} = t_{\leq n} \mid |T| < \infty\right\}.$$

4. [The effect of size-biased sampling.]
   A size-biased sample from $x_1, \ldots, x_n$ is a random variable I with $\mathbf{P}\{I = i\} \propto x_i$. This question is about the distributional properties of a size-biased sample from a collection of iid random variables.

Let $X_1, \ldots, X_n$ be iid positive random variables (where $n \geq 3$), and let S_n be their sum. Let J be a size-biased sample from $X_1, \ldots, X_n$, so

$$\mathbf{P}\{J = j \mid X_1, \ldots, X_n\} = X_j / S_n.$$

This determines the joint distribution of J and $X_1, \ldots, X_n$.

Next, for $1 \leq i \leq n-1$, let

$$X_i^* = \begin{cases} X_i & \text{if } i < J \\ X_{i+1} & \text{if } i \geq J, \end{cases}$$

so that $(X_1^*, \ldots, X_{n-1}^*) = (X_1, \ldots, X_{J-1}, X_{J+1}, \ldots, X_n)$ is "$(X_1, \ldots, X_n)$ with a size-biased sample removed." Let $S_{n-1}^* = X_1^* + \ldots + X_{n-1}^* = S_n - X_J$.

(a) Prove that X_J is conditionally independent of $(X_1^*, \ldots, X_{n-1}^*)$ given S_{n-1}^*, in that for any bounded Borel functions $f : \mathbb{R}^{n-1} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$,

$$\mathbf{E}\left[f(X_1^*, \ldots, X_{n-1}^*) g(X_J) \mid S_{n-1}^*\right] = \mathbf{E}\left[f(X_1^*, \ldots, X_{n-1}^*) \mid S_{n-1}^*\right] \mathbf{E}\left[g(X_J) \mid S_{n-1}^*\right].$$

(b) Prove that the conditional distribution of $(X_1^*, \ldots, X_{n-1}^*)$ given S_{n-1}^* is equivalent to the conditional distribution of $(X_1, \ldots, X_{n-1})$ given $X_1 + \ldots + X_{n-1}$, in that for any bounded Borel function $f : \mathbb{R}^{n-1} \to \mathbb{R}$,

$$\mathbf{E}\left[f(X_1^*, \ldots, X_{n-1}^*) \mid S_{n-1}^*\right] \stackrel{\mathrm{d}}{=} \mathbf{E}\left[f(X_1, \ldots, X_{n-1}) \mid S_{n-1}\right].$$

(Hint: Use the definition of conditional expectation.)