Math 589 – Winter 2020 – Assignment 1

Assigned on January 15, 2020.
Due on January 24, 2019 at 2 PM (by email or in my office)

1. Exercise 11.6 from the course notes.

   Proof. This is Theorem 4.6.3 from the course text (Durrett, Probability: Theory and Examples, Volume 5.)

2. Exercises 12.2,12.3,12.4,12.5,12.6 (a)-(c),12.8 from the course notes.

   Proof 12.2. Fix any Borel set $B \in \mathcal{B}(\mathbb{R})$ and let $T = \inf\{n : R_n \in B\}$. Then for any $n \geq 0$ we may write the event $\{T \leq n\}$ as

   $$\{T \leq n\} = \bigcup_{m=0}^{n} \{R_m \in B\}.$$

   For each $m \geq 0$ we have $\{R_m \in B\} \in \mathcal{F}_m \subset \mathcal{F}_n$, so as $\mathcal{F}_n$ is closed under finite unions it follows that $\{T \leq n\} \in \mathcal{F}_n$. All three assertions of the exercise follow by taking $B = (-\infty, 0)$ for T_0, $B = (1000, \infty)$ for T_1, and $B = \mathbb{R} \setminus ((-\infty, 0) \cup (1000, \infty))$ for T^*.

   Proof 12.3. First,

   $$\{T_1 \leq T_0\} \cap \{T_0 \leq n\} = \bigcup_{m=0}^{n} \{T_0 = m\} \cap \{T_1 \leq m\}.$$

   The events $\{T_0 = m\}$ and $\{T_1 \leq m\}$ both lie in $\mathcal{F}_m \subset \mathcal{F}_n$, and it follows by closure of $\mathcal{F}_n$ under finite unions that $\{T_1 \leq T_0\} \cap \{T_0 \leq n\} \in \mathcal{F}_n$; therefore $\{T_1 \leq T_0\}$ is in $\mathcal{F}_{T_0}$.

   Next,

   $$\{T_1 \leq T_0\} \cap \{T_1 \leq n\} = \bigcup_{m=0}^{n} \{T_1 = m\} \cap \{T_0 \leq m\}.$$

   The events $\{T_1 = m\}$ and $\{T_0 \leq m\}$ both lie in $\mathcal{F}_m \subset \mathcal{F}_n$, and it follows by closure of $\mathcal{F}_n$ under finite unions that $\{T_1 \leq T_0\} \cap \{T_1 \leq n\} \in \mathcal{F}_n$; therefore $\{T_1 \leq T_0\}$ is in $\mathcal{F}_{T_1}$.

   Finally, we write

   $$\{T_1 \leq T_0\} \cap \{T^* \leq n\} = \bigcup_{m=0}^{n} \{T_1 = m\} \cap \{T_0 \geq m\}.$$

   It follows just as before that $\{T_1 \leq T_0\} \cap \{T^* \leq n\} \in \mathcal{F}_n$; therefore $\{T_1 \leq T_0\}$ is in $\mathcal{F}_{T^*}$.

3. Let $(X_n, n \geq 1)$ be independent random variables in $L_2(\Omega, \mathcal{F}, \mathbf{P})$ with $\mathbf{E}[X_n] = 0$ and $\mathbf{E}\left[X^2\right] = \sigma^2 \in (0, \infty)$ for all n. Set $M_n = (\sum_{i=1}^n X_i)^2 - n\sigma^2$ for $n \geq 0$. Show that $(M_n, n \geq 0)$ is a martingale with respect to the natural filtration.

Proof 12.4.

Suppose the sequence $(x_n, n \geq 0)$ converges to $x \in \mathbb{R} \cup \{-\infty, \infty\}$. For any rationals $a < b$, if $b \leq x$ then $x_n > a$ for all n sufficiently large so the number of upcrossings of $[a, b]$ by $(x_n, n \geq 0)$ is finite. Similarly, if $a \geq x$ then $x_n < b$ for all n sufficiently large, so the number of upcrossings of $[a, b]$ by $(x_n, n \geq 0)$ is again finite. Finally, if $a < x < b$ then $x_n \in (a, b)$ for all n sufficiently large, which means that the number of upcrossings of $[a, b]$ by $(x_n, n \geq 0)$ is finite in this case as well.

Next, suppose that there exist rationals $a < b$ such that the number of upcrossings of $[a, b]$ by $(x_n, n \geq 0)$ is finite. Then there exists a strictly increasing sequence of integers $(n_i, i \geq 0)$ such that $x_{n_i} \leq a$ for $i \geq 0$; it follows $\liminf_{n\to\infty} x_n \leq \liminf_{i\to\infty} x_{n_i} \leq a$. Similarly, there exists a strictly increasing sequence of integers $(m_i, i \geq 0)$ such that $x_{m_i} \geq b$ for $i \geq 0$ it follows that $\limsup_{n\to\infty} x_n \leq \limsup_{i\to\infty} x_{m_i} \leq b$. Thus $(x_n, n \geq 0)$ does not converge.

Proof 12.5. We are given a sequence $(X_n, n \geq 1)$ of IID non-negative mean-one random variables in $L_1(\Omega, \mathcal{F}, \mathbf{P})$, and are told that $\log X_1 \notin L_1(\Omega, \mathcal{F}, \mathbf{P})$.

We write

$$\log X_i = P_i - N_i$$

where $P_i = (\log(X_i)^+ = \max(X_i, 0)$ and $N_i = (\log(X_i))^- = -\min(\log(X_i), 0)$. Note that

$$\log \prod_{i=1}^n X_i = \sum_{i=1}^n P_i - \sum_{i=1}^n N_i.$$

By Jensen's inequality,

$$\begin{aligned}\mathbf{E}\left[(\log(X_1))^+\right] &= \mathbf{E}\left[\max(\log(X_1), 0)\right] = \mathbf{E}\left[\log(\max(X_1, 1))\right] \\ &\leq \log \mathbf{E}\left[\max\right](X_1, 1) \leq 1 + \mathbf{E}\left[X\right]_1 = 2.\end{aligned}$$

Thus $P_i = \log(X_1))^+ \in L_1(\Omega, \mathcal{F}, \mathbf{P})$, and by the law of large numbers, it follows that

$$n^{-1} \sum_{i=1}^n P_i \overset{\text{a.s.}}{\to} \mathbf{E}\left[P_1\right] < \infty.$$

Since $\log X_1$ is not in $L_1(\Omega, \mathcal{F}, \mathbf{P})$ by assumption, it must be that $\mathbf{E}N_i = \infty$. By the monotone convergence theorem this implies that for any $K > 0$ there exists $M > 0$ such that $\mathbf{E}N_i^{\leq M} \geq K$. The random variables $N_i^{\leq M}$ are bounded so in L_1, and it follows that

$$n^{-1} \sum_{i=1}^n N_i^{\leq M} \to \mathbf{E}\left[N_1^{\leq M}\right] \geq K.$$

This implies that almost surely

$$\begin{aligned}\limsup_{n\to\infty} n^{-1} \log \prod_{i=1}^n X_i &\leq \limsup_{n\to\infty} n^{-1} \sum_{i=1}^n P_i - \liminf_{n\to\infty} n^{-1} \sum_{i=1}^n N_i \\ &\leq \limsup_{n\to\infty} n^{-1} \sum_{i=1}^n P_i - \liminf_{n\to\infty} n^{-1} \sum_{i=1}^n N_i^{\leq M} \\ &\overset{\text{a.s.}}{=} \mathbf{E}\left[P_1\right] - \mathbf{E}\left[N_1^{\leq M}\right] \leq \mathbf{E}\left[P_1\right] - K\,.\end{aligned}$$

Since $K > 0$ was arbitrary it follows that $n^{-1} \log \prod_{i=1}^{n} X_i \to -\infty$ almost surely.

Proof 12.6.

I will write $\tau = \inf\{n \geq 0 : X_n \in S\}$; in one version of notes this was called T but in the part of the current question T has a different meaning so there was a bit of notational overload (sorry)!

(a) Let $Z = \{z \in V : b(z) = 0\}$. Suppose Z is non-empty and fix any state $u \in Z$. Then

$$b(u) = \sum_{v \in V} P(u,v) b(v).$$

So it must be that $P(u,v) = 0$ for all v with $b(v) > 0$, i.e. for all v with $v \in V \setminus Z$. We have shown that there is no pair u, v with $u \in Z$ and $v \in V \setminus z$ such that $P(u,v) > 0$; therefore the chain is not irreducible.

(b) Write $b_{\max} = \max(b(w) : w \in V)$. Suppose $T \cap S$ is empty. For any node $u \in T$ we have

$$b_{\max} = b(u) = \sum_{v \in V} P(u,v) b(v) = b_{\max} - \sum_{v \in V} P(u,v)(b(v) - b_{\max}).$$

It follows that $b(v) = b_{\max}$ for all nodes v with $P(u,v) > 0$. This implies that all nodes v with $P(u,v) > 0$ have $v \in T$.

We claim that it follows that for all $u \in T$, $\mathbf{P}\{\tau = \infty \mid X_0 = u\} = 1$. To see this, simply note that for $u \in T$, since $P(u,v) = 0$ for $v \notin T$, we have

$$\begin{aligned}
\mathbf{P}\{\tau > n \mid X_0 = u\} &= \sum_{v \in T} \mathbf{P}\{\tau > n, X_1 = v \mid X_0 = u\} \\
&= \sum_{v \in T} \mathbf{P}\{\tau > n \mid X_1 = v, X_0 = u\} \mathbf{P}\{X_1 = v \mid X_0 = u\} \\
&= \sum_{v \in T} \mathbf{P}\{\tau > n - 1 \mid X_0 = v\} P(u,v).
\end{aligned}$$

It follows by induction that $\mathbf{P}\{\tau > n \mid X_0 = u\} = 1$ for all $u \in T$ and $n > 0$. The dominated convergence theorem then implies that $\mathbf{P}\{\tau = \infty \mid X_0 = u\} = 1$ for all $u \in T$.

But by definition,

$$b(u) = \mathbf{E}\left[X_\tau \mathbf{1}_{[\tau < \infty]} \mid X_0 = u\right];$$

for $u \in T$, since $\mathbf{P}\{\tau > n \mid X_0 = u\} = 0$ it follows that $b(u) = 0$, a contradiction.

(c) Suppose that $R := T \setminus S$ is non-empty, and let $Q = (V \setminus S) \setminus R$ be the part of $(V \setminus S)$ which is not in R. States $u \in R$ have $b(u) = b_{\max}$ and states in $u \in Q$ have $b(u) < b_{\max}$. If Q is non-empty then by assumption, there exist $u \in R$ and $v \in Q$ such that $P(u,v) > 0$. But then

$$\begin{aligned}
b(u) &= \sum_{w \in V} P(u,w) b(w) \\
&\leq (\sum_{w \in V} P(u,w) b_{\max}) - P(u,v)(b_{\max} - b(v)) \\
&= b_{\max} - P(u,v)(b_{\max} - b(v)) < b_{\max}.
\end{aligned}$$

This contradicts that $u \in T$. It follows that Q is empty. Since $Q \cup R = V \setminus S$, this implies that $b_{V \setminus S}$ is a constant function (taking only the value $b_{\max}$).

4. [Filtrations and changes of measure] Let $(\Omega, \mathcal{F})$ be a σ-algebra, let $\mathbf{P}$,$\mathbf{Q}$ be two probability measures on $(\Omega, \mathcal{F})$, and write $\mathbb{E}_{\mathbf{P}}$ and $\mathbb{E}_{\mathbf{Q}}$ for the corresponding expectation operators.

   Fix an increasing sequence of sub-σ-algebras $(\mathcal{F}_n)_{n\geq 1}$ with $\sigma(\bigcup_n \mathcal{F}_n) = \mathcal{F}$. Write $\mathbf{P}_n := \mathbf{P}|_{\mathcal{F}_n}$ and $\mathbf{Q}_n := \mathbf{Q}|_{\mathcal{F}_n}$. Suppose that $\mathbf{Q}_n \ll \mathbf{P}_n$ for all n, and write $X_n = d\mathbf{Q}_n/d\mathbf{P}_n : \Omega \to [0, \infty)$ for the corresponding Radon-Nikodym derivatives. Prove that X_n is an $\mathcal{F}_n$-martingale with respect to $\mathbf{E}_{\mathbf{P}}$.

   Proof. For all $n \geq 0$, by the definition of the Radon-Nikodým derivative, X_n is $\mathcal{F}_n/\mathcal{B}(\mathbb{R})$-measurable and for all $E \in \mathcal{F}_n$,

$$\mathbf{E}_{\mathbf{P}}(X_n \mathbf{1}_{[E]}) = \int_E X_n \mathrm{d}\mathbf{P} = \int_E 1 \mathrm{d}\mathbf{Q} = \mathbf{Q}(E).$$

   In particular, $\mathbf{E}(X_n) = \mathbf{Q}(\Omega) = 1$ so $X_n \in L_1(\Omega, \mathcal{F}_n, \mathbf{P})$.

   Next, for $n \geq 1$, since $\mathcal{F}_{n-1} \subset \mathcal{F}_n$, the preceding displayed identity implies that for all $E \in \mathcal{F}_{n-1}$,

$$\mathbf{E}_{\mathbf{P}}(X_n \mathbf{1}_{[E]}) = \mathbf{Q}(E) = \mathbf{E}_{\mathbf{P}}(X_{n-1} \mathbf{1}_{[E]}) \,.$$

   It follows by the definition of conditional expectation that X_{n-1} is a version of $\mathbf{E}_{\mathbf{P}}(X_n \mid \mathcal{F}_{n-1})$.