Math 589 – Winter 2020 – Assignment 1

Assigned on January 15, 2020.
Due on January 24, 2019 at 2 PM (by email or in my office)

1. Exercise 11.6 from the course notes.

2. Exercises 12.2,12.3,12.4,12.5,12.6 (a)-(c),12.8 from the course notes.

3. Let $(X_n, n \geq 1)$ be independent random variables in $L_2(\Omega, \mathcal{F}, \mathbf{P})$ with $\mathbf{E}[X_n] = 0$ and $\mathbf{E}\left[X^2\right] = \sigma^2 \in (0, \infty)$ for all n. Set $M_n = (\sum_{i=1}^n X_i)^2 - n\sigma^2$ for $n \geq 0$. Show that $(M_n, n \geq 0)$ is a martingale with respect to the natural filtration.

4. [Filtrations and changes of measure] Let $(\Omega, \mathcal{F})$ be a σ-algebra, let $\mathbf{P}$,$\mathbf{Q}$ be two probability measures on $(\Omega, \mathcal{F})$, and write $\mathbb{E}_{\mathbf{P}}$ and $\mathbb{E}_{\mathbf{Q}}$ for the corresponding expectation operators.

   Fix an increasing sequence of sub-σ-algebras $(\mathcal{F}_n)_{n\geq 1}$ with $\sigma(\bigcup_n \mathcal{F}_n) = \mathcal{F}$. Write $\mathbf{P}_n := \mathbf{P}|_{\mathcal{F}_n}$ and $\mathbf{Q}_n := \mathbf{Q}|_{\mathcal{F}_n}$. Suppose that $\mathbf{Q}_n \ll \mathbf{P}_n$ for all n, and write $X_n = d\mathbf{Q}_n/d\mathbf{P}_n : \Omega \to [0, \infty)$ for the corresponding Radon-Nikodym derivatives. Prove that X_n is an $\mathcal{F}_n$-martingale with respect to $\mathbf{E}_{\mathbf{P}}$.