Math 587
Midterm Solutions
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(i) The collection \( \mathcal{A} = \{ E \subset \mathbb{N} : |E| < \infty \text{ or } |\mathbb{N} \setminus E| < \infty \} \) is one possibility; there are uncountably many others.

(ii) Write \( \mathcal{P} = \{ A \supseteq \mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-field} \} \) and \( \mathcal{Q} = \{ A \supseteq B : A \text{ is a } \sigma\text{-field} \} \). Then \( \mathcal{Q} \subset \mathcal{P} \) since \( A \subset B \), so

\[
\sigma(A) = \bigcap_{A \in \mathcal{P}} \mathcal{A} \subset \bigcap_{A \in \mathcal{Q}} \mathcal{A} = \sigma(B).
\]

On the other hand, \( \sigma(A) \in \mathcal{Q} \) by assumption, so \( \sigma(B) = \bigcap_{A \in \mathcal{Q}} \mathcal{A} \subseteq \sigma(A) \). \( \square \)

(iii) Let \( G = \bigcup_{J \subset I, J \text{ countable}} \sigma(X_j, j \in J) \), we wish to show that \( G = \sigma(\{X_i, i \in I\}) \).

Recall that \( \sigma(\{X_i, i \in I\}) = \sigma(\bigcup_{i \in I} \sigma(X_i)) = \bigcap_{R \in \mathcal{R}} \mathcal{R} \)

where \( \mathcal{R} = \{ \mathcal{F} : \mathcal{F} \text{ a } \sigma\text{-field over } \Omega \text{ s.t. } X_i \text{ is } \mathcal{F}/B(\mathbb{R})\text{-measurable} \} \).

It is immediate that for all \( J \subset I \), \( \sigma(X_j, j \in J) \subseteq \sigma(X_i, i \in I) \), so \( G \subseteq \sigma(X_i, i \in I) \).

On the other hand, \( \sigma(X_i) \subset G \) for all \( i \in I \). We claim \( G \) is a \( \sigma\)-field; if so, then \( G \in \mathcal{R} \) so \( \sigma(X_i, i \in I) = \bigcap_{R \in \mathcal{R}} \mathcal{R} \subset G \).

To see that \( G \) is a \( \sigma\)-field, argue as follows:

(i) \( G \) clearly contains \( \Omega \) (since each \( \sigma(X_j, j \in J) \) does).

(ii) If \( E \in G \) then \( E \subset \sigma(X_j, j \in J) \) for some \( J \subset I \) countable.

Then \( E^c \subset \sigma(X_j, j \in J) \) so \( E^c \in G \).

(iii) If \( (E_n, n \geq 1) \in G \) then there exist countable sets \( (J_n, n \geq 1) \), all subsets of \( I \), such that \( E_n \in \sigma(X_j, j \in J_n) \) for each \( n \geq 1 \).

Taking \( J = \bigcup_{n \geq 1} J_n \), then for all \( n \geq 1 \), \( \sigma(X_j, j \in J) \subseteq \sigma(X_j, j \in J) \), so \( E_n \in \sigma(X_j, j \in J) \).

It follows that \( \bigcup_{n \geq 1} E_n \in \sigma(X_j, j \in J) \); since \( J \) is countable this gives that \( \bigcup_{n \geq 1} E_n \in G \). \( \square \)
Write \( \mathcal{F}_i = \sigma(1_{A_i}) \) and note that \( \mathcal{F}_i = \{ \emptyset, A_i, A_i^c, \Omega \} \).

By definition, \( 1_{A_i}, \ldots, 1_{A_n} \) are mutually independent if and only if \( \mathcal{F}_i, \ldots, \mathcal{F}_n \) are mutually independent. If this occurs then \( A_1, \ldots, A_n \) are mutually independent since \( A_i \in \mathcal{F}_i \) for each \( i \in \{n\} \).

To prove the converse, we show by induction on \( n \) that if \( A_1, \ldots, A_n \) are mutually independent then \( 1_{A_1}, \ldots, 1_{A_n} \) are mutually independent.

The base case \( n = 1 \) is trivial. Suppose the claim holds for \( 1 < n' < n \), and suppose that \( A_1, \ldots, A_n \) are mutually independent. We wish to show that for any events \( E_i \in \mathcal{F}_i, \ldots, E_n \in \mathcal{F}_n \), it holds that

\[
\Pr(E_1 \cap \cdots \cap E_n) = \prod_{i=1}^{n} \Pr(E_i).
\]

If any of the \( E_j \) equal \( \emptyset \) then both sides are 0 so the equality holds. If any \( E_j \) equals \( \Omega \) then we may conclude by applying the induction hypothesis to the collection \( (E_i, i \in [n] \setminus \{j\}) \), which has size \( n-1 \). It remains to handle the cases when either \( E_i = A_i \) or \( E_i = A_i^c \) for each \( i \in [n] \). For this, simply observe that for any \( j \in [n] \),

\[
\Pr(\bigcap_{i \in [n]} A_i) + \Pr(A_j^c \cap \bigcap_{i \in [n] \setminus \{j\}} A_i) = \Pr(\bigcap_{i \in [n]} A_i)
\]

by additivity. But since \( A_1, \ldots, A_n \) are mutually independent,

\[
\Pr(\bigcap_{i \in [n]} A_i) = \prod_{i \in [n]} \Pr(A_i) \quad \text{and} \quad \Pr(\bigcap_{i \in [n]} A_i) = \prod_{i = 1}^{n} \Pr(A_i),
\]

so the preceding equation gives

\[
\Pr(A_j^c \cap \bigcap_{i \in [n] \setminus \{j\}} A_i) = (1 - \Pr(A_j)) \cdot \prod_{i \in [n] \setminus \{j\}} \Pr(A_i)
\]

We may repeat the same argument for other indices, so it follows that for any \( J \subseteq [n] \),

\[
\Pr(\bigcap_{j \in J} A_j^c \cap \bigcap_{i \in [n] \setminus J} A_i) = \prod_{j \in J} (1 - \Pr(A_j)) \cdot \prod_{i \in [n] \setminus J} \Pr(A_i).
\]
If you remembered that "any finite collection of information is unimportant for T" you should have arrived at something like this.

(ii) Let \((X_n, n \geq 1)\) be independent random variables and let \(T\) be their tail \(\sigma\)-algebra. Then for any event \(E \subseteq T\) either \(P(E) = 0\) or \(P(E) = 1\).

(iii) Let \(X'_n = -X_n\) for \(n \geq 1\). Then \((X'_n, n \geq 1)\) are again indep.
With \(P(X'_1 = 1) = \frac{1}{2} = P(X'_1 = -1)\), so setting \(S'_n = \sum_{i=1}^{\infty} X'_i\), we have
\[
P\left(\limsup_{n \to \infty} S'_n > -\infty\right) = P\left(\limsup_{n \to \infty} S_n > -\infty\right)
\]
But if \(\limsup_{n \to \infty} S_n = -\infty\), then
\[
\limsup_{n \to \infty} S'_n \geq \liminf_{n \to \infty} S_n = \liminf_{n \to \infty} (-S_n) = -\limsup_{n \to \infty} S_n = \infty > -\infty,
\]
so by subadditivity
\[
1 = P\left(\limsup_{n \to \infty} S'_n > -\infty \text{ or } \limsup_{n \to \infty} S_n > -\infty\right)
\leq P\left(\limsup_{n \to \infty} S'_n > -\infty\right) + P\left(\limsup_{n \to \infty} S_n > -\infty\right) = 2P\left(\limsup_{n \to \infty} S_n > -\infty\right)
\]
So \(P\left(\limsup_{n \to \infty} S_n > -\infty\right) \geq \frac{1}{2}\).

But \(\{\limsup_{n \to \infty} S_n > -\infty\}\) is a tail event, so has probability either 0 or 1; thus \(P\left(\limsup_{n \to \infty} S_n > -\infty\right) = 1\).
Next, write $p_k = P(\limsup S_n = k)$. Then

$$\sum_{k \in \mathbb{Z}} p_k = P(\limsup S_n < \infty) \in [0, 1],$$

so there is some $k \in \mathbb{Z}$ for which $p_k$ is maximized.

But note that with $Y_n = X_{n+1}$ for $n \geq 1$ then $(Y_n, n \geq 1)$ have the same joint distribution as $(X_n, n \geq 1)$, so

$$p_k = P(\limsup S_n^* = k).$$

But note that $\limsup S_n = X_{i+1} \limsup S_n^* = X_{i+1}$, and $X_{i+1}$ and $\limsup S_n^*$ are independent, so

$$p_k = P(\limsup S_n = k) = P(\limsup S_n^* = k - 1) + P(\limsup S_n^* = k + 1) = \frac{1}{2} P(\limsup S_n^* = k - 1) + \frac{1}{2} P(\limsup S_n^* = k + 1) = \frac{1}{2} p_{k-1} + \frac{1}{2} p_{k+1}$$

If $p_k = \frac{p_{k-1} + p_{k+1}}{2}$ then either $p_{k+1} \geq p_k$ or $p_{k-1} \geq p_k$.

But $p_k$ is maximal! So we must have $p_k = p_{k-1} = p_{k+1}$.

It follows by induction that $p_k = p_k$ for all $k \in \mathbb{Z}$; since

$$\sum_{k \in \mathbb{Z}} p_k \leq 1$$

it must therefore be that $p_k = 0$ for all $k \in \mathbb{Z}$.

Thus $P(\limsup S_n = \infty) = \sum_{k \in \mathbb{Z}} P(\limsup S_n = k) - P(\limsup S_n = -\infty) = 1 - 0 = 1$. □