Math 587 – MIDTERM

Tuesday, Oct 22, 2019, 14:35-15:55

Relax and take a few deep breaths before continuing.
Read questions carefully before answering.
In your responses, state any result you use from class, and try to write clearly (in terms of both handwriting and explanation).
There are a total of 30 points possible.

1. (i) **4 points.** Describe a field over $\mathbb{N}$ that is not a σ-field.

   (ii) **4 points.** Suppose that A, B are collections of subsets of Ω and $A \subset B \subset \sigma(A)$. Prove that $\sigma(A) = \sigma(B)$.

   (iii) **4 points.** Fix an arbitrary index set I, and let $(X_i, i \in I)$ be functions defined on a common space Ω; that is, $X_i : \Omega \to \mathbb{R}$ for all $i \in I$. Show that

$$\sigma(X_i, i \in I) = \bigcup_{J \subset I, J \text{ countable}} \sigma(X_j, j \in J)\,.$$

2. **4 points.** Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $A_1, \ldots, A_n$ be elements of $\mathcal{F}$. Prove that the following two statements are equivalent.

   (a) The events $A_1, \ldots, A_n$ are mutually independent.

   (b) The indicator random variables $\mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_n} : \Omega \to \mathbb{R}$ are mutually independent.

3. (i) **4 points** Let $(\mathcal{F}_n, n \geq 1)$ be a sequence of σ-algebras over a set Ω. Define the tail σ-algebra $\mathcal{T}$ of the sequence.

   (ii) **4 points** State Kolmogorov's $0-1$ law.

   (iii) **6 points** Let $(X_n, n \geq 1)$ be independent random variables defined on a common space, such that for all n,

$$X_n = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2\,. \end{cases}$$

   Then set $S_n = \sum_{i=1}^n X_i$. Prove that $\mathbf{P}(S_n \geq 0 \text{ infinitely often}) = 1$. (Hint: first show that $\mathbf{P}(\limsup_{n\to\infty} S_n > -\infty) = 1$; this will get you half the points.)