1. Measure Theory

**Def**: Fix a set $\Omega$ and a set $\mathcal{A}$ of subsets of $\Omega$ with $\emptyset \in \mathcal{A}$.

$\mathcal{A}$ is a ring if

a) If $E, F \in \mathcal{A}$ then $E \cup F \in \mathcal{A}$,

b) If $E, F \in \mathcal{A}$ then $E \setminus F \in \mathcal{A}$.

$\mathcal{A}$ is a $\pi$-system if

a) If $E, F \in \mathcal{A}$ then $E \cap F \in \mathcal{A}$,

b) If $E \in \mathcal{A}$ then $E^c \in \mathcal{A}$.

$\mathcal{A}$ is a field if it is a ring and also

b) If $E \in \mathcal{A}$ then $E^c \in \mathcal{A}$.

Exercise: Fields are $\pi$-systems.

$\mathcal{A}$ is a $\sigma$-field if it is a field and also

a') For any seq. $(A_n, n \geq 1)$ of elements of $\mathcal{A}$, $\bigcup_{n \geq 1} A_n \in \mathcal{A}$.

**Def**: For any set $\mathcal{A}$ of subsets of $\Omega$, the $\sigma$-field generated by $\mathcal{A}$ is

$$\sigma(\mathcal{A}) := \bigcap \{ \mathcal{F} : \mathcal{F} \text{ a } \sigma\text{-field} \}.$$
The definition in pictures

\[ A \rightarrow \sigma(A) = \bigcup \{ \mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-field}\} \]

Pre-measure \( \mu \) on ring \( A \):
1. \( \mu(\emptyset) = 0 \);
2. If \( (A_n)_{n \geq 1} \) disjoint elements of \( A \), then \( \bigcup_{n \geq 1} A_n \) then \( \sum_{n \geq 1} \mu(A_n) = \mu(\bigcup_{n \geq 1} A_n) \)

\[ \begin{align*}
\text{Additive:} & \quad \mu(\emptyset \cap F) = 0 \Rightarrow \mu(E \cup F) = \mu(E) + \mu(F) \\
\text{Countably Additive:} & \quad \mu \left( \bigcup_{n \geq 1} E_n \right) = \sum_{n \geq 1} \mu(E_n)
\end{align*} \]
Building measures

**Def** Fix a ring \( A \) over \( \Omega \). A pre-measure on \( A \) is a function \( \mu : A \to [0, \infty] \) with \( \mu(\emptyset) = 0 \) st.

for any seq. \( (A_n, n \geq 1) \) of disjoint elements of \( A \),

if \( \bigcup_{n=1}^{\infty} A_n \in A \) then \( \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) \)

We then say \( (\Omega, A, \mu) \) is a pre-measure space.

**Carathéodory Extension Theorem** Let \( (\Omega, A, \mu) \) be a pre-measure space. Then there exists a \( \sigma \)-field \( \mathcal{F} \) containing \( A \) st. \( \mu \) extends to a measure on \( \mathcal{F} \).

**Dynkin's Theorem** Let \( (\Omega, \mathcal{F}) \) be a set with a \( \sigma \)-field on it, and let \( \mathcal{P} \subset \mathcal{F} \) be a \( \pi \)-system with \( \sigma(\mathcal{P}) = \mathcal{F} \). If \( \mu_1, \mu_2 \) are measures on \( \mathcal{F} \) and \( \mu_1(E) = \mu_2(E) \) for all \( E \in \mathcal{P} \) then \( \mu_1 \equiv \mu_2 \).
Aside

It seems to me that the following should be true.

Let \((\mathcal{M}, \mathcal{P})\) be a set with a \(\pi\)-system on it. Let \(\mu : \mathcal{P} \to [0, \infty)\) be s.t. \(\mu(\emptyset) = 0\) and if \((P_n, n \geq 1)\) are disjoint elements of \(\mathcal{P}\) s.t. \(\bigcup_{n \geq 1} P_n \subseteq Q \in \mathcal{P}\)

\[ \text{then } \sum_{n \geq 1} \mu(P_n) \leq \mu(Q) \]

[Note: If \(Q \subseteq \bigcup_{n \geq 1} P_n\), then \(Q = \bigcup_{n \geq 1} Q \cap P_n\)]

Then there exists a measure on \(\sigma(\mathcal{P})\) extending \(\mu\).
Examples

- \( \Omega = \mathbb{R} \), \( A = \) Finite unions of intervals \( (a,b] = \mathbb{Z} (a_i, b_i) u \ldots u (a_k, b_k) \)

\[ (\text{CDF}: F_x(b) - F_x(a) = \mathbb{P}(x \in (a,b]) = \mu_x(a,b)] \]

\( a_i, \ldots, a_k, b_i, \ldots, b_k \in \mathbb{R}^2 \)

\( A \) is an algebra; want to know that \( F_x \) determines dist. of \( X \).

- \( \Omega = \{0,1\}^\mathbb{N} = \{(x_i, i \geq 1) : x_i \in \{0,1\} \} \).

\( A = \) Cylinder sets. Cylinder set: for \( s \in \mathbb{N} \) finite and \( y = (y_i, i \in s) \),

\[ C_y = \{ x \in \Omega : x_i = y_i \ \forall i \in s \} \]

For cylinder set \( C_y \) set \( \mu(C_y) = \left( \frac{1}{2} \right)^{|s|} \) ("IID Fair coins")

Should be able to extend \( \mu \) to a p.m. on \((\Omega, \sigma(A))\); \( \mu \) models "an \( \infty \) sequence of fair coin tosses".
Carathéodory Proof
Idea: Approximate from above.

Let $(\Omega, \mathcal{A}, \mu)$ be a pre-measurable space. For $B \subseteq \Omega$ let

$\mu^*(B) = \inf \left( \sum_{n=1}^{\infty} \mu(A_n) : A_n \subseteq \mathcal{A}, \bigcup_{n=1}^{\infty} A_n \subseteq B \right)$

Prop: $\mu^*$ is an outer measure: $\mu^* : 2^{\Omega} \to [0, \infty]$ satisfies

i) $\mu^*(\emptyset) = 0$;
ii) $E \subseteq F \implies \mu(E) \leq \mu(F)$;
iii) if $(E_i, i \geq 1)$ are subsets of $\Omega$ then $\mu^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.

Def: Given an outer measure $\mu^*$ on $\Omega$, say $A \subseteq \Omega$ is $\mu^*$-additive if for all $B \subseteq \Omega$, $\mu^*(B) = \mu^*(A \cap B) + \mu^*(A^c \cap B)$.

Carathéodory Lemma Let $\mathcal{F} = \{ A \subseteq \Omega : A \text{ is } \mu^* \text{-additive} \}$

Define $\mu : \mathcal{F} \to [0, \infty]$ by $\mu(B) = \mu^*(B)$. Then $(\Omega, \mathcal{F}, \mu)$ is a measure space (i.e. $\mathcal{F}$ is a $\sigma$-algebra over $\Omega$ and $\mu$ is a measure on $\mathcal{F}$).
(i) If \( B \in A \) then \( \mu^*(B) \leq \mu(B) \) since \( (B, \phi, \phi, \ldots) \) covers \( B \). In particular \( \mu^*(\phi) \leq \mu(\phi) = 0 \) so \( \mu^*(\phi) = 0 \).

(ii) If \( E \subseteq F \) then any cover of \( F \) is a cover of \( E \) so \( \mu^*(F) \) is an inf over a smaller set so \( \mu^*(F) \geq \mu^*(E) \).

(iii) "Dyadic trick." Given \( (E_i, i \geq 1) \) subsets of \( \Omega \). Write \( E = \bigcup_{i \geq 1} E_i \).

We prove: \( \forall \varepsilon > 0, \mu^*(E) \leq \left( \sum_{i \geq 1} \mu^*(E_i) \right) + \varepsilon \).

Fix \( \varepsilon > 0 \), then for all \( i \geq 1 \), fix a cover \( (A_n^i, n \geq 1) \) of \( E_i \) s.t. \( \sum_{n \geq 1} \mu(A_n^i) \leq \mu^*(E_i) + \frac{\varepsilon}{2^i} \).

Then \( (A_n^i, n, i \geq 1) \) covers \( E \) so \( \mu^*(E) \leq \sum_{n \geq 1} \mu(A_n^i) \leq \sum_{i \geq 1} \left( \mu^*(E_i) + \frac{\varepsilon}{2^i} \right) = \left( \sum_{i \geq 1} \mu^*(E_i) \right) + \varepsilon \). \( \square \)
Proof of Carathéodory Lemma

Step 1: prove $\mathcal{F}$ is a $\sigma$-field

Step 2: Prove $\mu$ is a measure on $\mathcal{F}$.

**Step 1:** $\mathcal{F}$ abv. closed under complements (def. is invariant to $A \mapsto A^c$)

**Closure under $\cap$ trickier.** Fix $A_1, A_2 \in \mathcal{F}$ and any $B \subseteq \Omega$.

Write $B = B_0 \cup B_1 \cup B_2 \cup B_{12}$ according to $\cap$ with $A_1, A_2$.

Then $A_1 \in \mathcal{F}$, $A_2 \in \mathcal{F}$, $B_{12} \in \mathcal{F}$, $B_1 \in \mathcal{F}$.

$$
\mu^*(B) = \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c)
$$

$$
\mu^*(B) = \mu^*(B_1) + \mu^*(B_{12}) + \mu^*(B_0) + \mu^*(B_2)
$$

Also

$$
\mu^*(B \setminus B_{12}) = \mu^*(B_1) + \mu^*(B_0) + \mu^*(B_2)
$$

So

$$
\mu^*(B) = \mu^*(B \setminus B_{12}) + \mu^*(B_{12}) = \mu^*(B \cap (A_1 \cap A_2^c)) + \mu^*(B \cap A_1 A_2)
$$
Countable $U$: Fix disjoint sets $(A_n, n \geq 1)$ in $\mathcal{B}$, let $A = \bigcup_{n=1}^{\infty} A_n$ Fix $B \in \mathcal{B}$. Since $\mu^*$ is an outer measure, it is subadditive so $\mu(A) \leq \mu(A \cap B) + \mu(A^c \cap B)$; need to prove $\geq$.

"Cut $A$ into pieces" with $A_1$, then $A_2$, etc. Disjointness plus the fact that all $A_i \subset A$ gives

$$
\mu^*(B) = \mu^*(A_1 \cap B) + \mu^*(A_2 \cap B) + \cdots + \mu^*(A_n \cap B) + \mu^*(B \cap \bigcap_{i=1}^{n} A_i^c)
$$

So

$$
\mu^*(B) \geq \mu^*(A_1 \cap B) + \cdots + \mu^*(A_n \cap B) + \mu^*(A^c \cap B)
$$

Take a limit in $n$ to get

$$
\mu^*(B) \geq \sum_{i=1}^{\infty} \mu^*(A_i \cap B) + \mu^*(A^c \cap B)
$$

$$
\geq \mu((\bigcup_{i=1}^{\infty} A_i) \cap B) + \mu^*(A^c \cap B) = \mu(A \cap B) + \mu(A^c \cap B),\text{ so } A \in \mathcal{B}.
$$
We also just proved that if \((A_n, n \geq 1)\) disjoint sets in \(\mathcal{F}\) then
\[
\mu^*(A) = \sum_{i \geq 1} \mu^*(A_i \cap A) + \mu^*(A^c \cap A) = \sum_{i \geq 1} \mu^*(A_i)
\]
so \(\mu^*(A) = \sum_{i \geq 1} \mu^*(A_i)\), i.e. \(\mu\) restricts to a measure on \(\mathcal{F}\).

Proof of Carathéodory Theorem

Let \(\mu^*\) be the outer measure as above.

**Step 1**: If \(A \in \mathcal{A}\) then \(\mu^*(A) = \mu(A)\)
(since \(\mu^*\) extends \(\mu\)).

**Step 2**: \(A \in \mathcal{F}\)
(since \(\mathcal{F}\) extends \(\mathcal{A}\)).

**Step 1**: We know \(\mu^*(A) \leq \mu(A)\), want rev. ineq.

Let \((A_i, i \geq 1)\) be a cover of \(A\). For \(n \geq 1\) let \(B_n = A_n \setminus (A_1 \cup \cdots \cup A_{n-1})\)

Then \((B_i, i \geq 1)\) is a disjoint cover of \(A\) (\(\bigcap_{i \geq 1} A_i = \bigcap_{i \geq 1} B_i \cup B_n\)), \(\bigcup_{i \geq 1} (B_i \cap A) = A\) so
\[
\mu(A) = \sum_{i \geq 1} \mu(B_i \cap A) \leq \sum_{i \geq 1} \mu(B_i) \leq \sum_{i \geq 1} \mu(A_i).
\]
Take inf over covers \((A_i, i \geq 1)\) of \(A\) to get
\[
\mu(A) \leq \mu^*(A) \quad \square
\]
Step 2: Need to show if \( A \subseteq A \), then \( A \) is \( \mu^* \)-additive: if \( B \subseteq \mathcal{S}_\mu \), \( \mu^*(B) = \mu^*(A \cap B) + \mu^*(A^c \cap B) \).

\[ \mu^*(B) = \mu^*(A \cap B) + \mu^*(A^c \cap B) \]

"\( \leq \)" easy (subadditivity)

"\( = \)" Fix \( \varepsilon > 0 \), fix a cover \((A_n, n \geq 1)\) of \( B \) with elements of \( A \)

\[ \exists \sum \mu(A_n) \leq \mu^*(B) + \varepsilon \]

Then \((A \cap A_n, n \geq 1)\) covers \( A \cap B \) and \((A^c \cap A_n, n \geq 1)\) covers \( A^c \cap B \), so

\[ \mu^*(A \cap B) + \mu^*(A^c \cap B) \leq \sum \mu^*(A \cap A_n) + \sum \mu^*(A^c \cap A_n) \]

\[ = \sum \mu^*(A \cap A_n) + \mu^*(A^c \cap A_n) \]

But \( \varepsilon > 0 \) arbitrary so

\[ \mu^*(A \cap B) + \mu^*(A^c \cap B) \leq \mu^*(B) + \varepsilon \]
Last class:

Carathéodory Extension Theorem Let $(\Omega, \mathcal{A}, \mu)$ be a pre-measure space. Then there exists a $\sigma$-field $\mathcal{F}$ containing $\mathcal{A}$ s.t. $\mu$ extends to a measure on $\mathcal{F}$.

The field $\mathcal{F}$ was the sets $E$ s.t. $\mu(F) = \mu^*(E \cap F) + \mu^*(E^c \cap F)$ for all $F \subset \Omega$.

If $\mathcal{A} = \mathcal{A}(\mathbb{R}) := \{(a,b] \cup \cdots : a_k, b_k \in \mathbb{N}, -\infty < a, b, \ldots < a_k < b_k < \infty \}$

then $\mathcal{F}$ is called the Lebesgue measurable sets of $\mathbb{R}$; denote this by $\mathcal{L}(\mathbb{R})$.

NB: $\mathcal{A}(\mathbb{R})$ is not the smallest $\sigma$-field containing $\mathcal{A}$. The smallest is $\sigma(\mathcal{A}(\mathbb{R})) = \mathcal{B}(\mathbb{R})$, the Borel $\sigma$-field over $\mathbb{R}$. In fact, $\mathcal{L}(\mathbb{R})$ is the completion of $\mathcal{B}(\mathbb{R})$.

This class:

1) Dynkin's Theorem: Let $(\Omega, \mathcal{F})$ be a set with a $\sigma$-field on it, and let $\mathcal{P} \subset \mathcal{F}$ be a $\pi$-system with $\sigma(\mathcal{P}) = \mathcal{F}$. If $\mu_1, \mu_2$ are measures on $\mathcal{F}$ and $\mu_1(E) = \mu_2(E)$ for all $E \in \mathcal{P}$ then $\mu_1 \equiv \mu_2$.

2) Stieltjes Measures
One more definition:
A set $A \subset 2^\Omega$ is a $\lambda$-system over $A$ if $\mathcal{A} \in \mathcal{A}$ and
- $E, F \in \mathcal{A}$, $E \subset F \Rightarrow F \setminus E \in \mathcal{A}$
- $E_n \in \mathcal{A}$, $n \geq 1$ with $E_n \cap E = E \in \mathcal{A}$

Exercises
① If $A$ is a $\sigma$-field over $\Omega$ then $A$ is a $\lambda$-system over $\Omega$.
② If $A \subset 2^\Omega$ is a $\pi$-system and a $\lambda$-system then $A$ is a $\sigma$-field.
③ If $\forall \mathcal{V}_i, i \in I$ are $\lambda$-systems over $\Omega$ then $\bigcap_{i \in I} \mathcal{V}_i$ is a $\lambda$-system over $\Omega$.

Dynkin's $\pi$-system lemma
Let $\mathcal{P}$ be a $\pi$-system over $\Omega$. Then
$$\left[ \sigma(\mathcal{P}) := \bigcap \{ \mathcal{P} : \mathcal{P} \text{ a } \sigma\text{-field over } \Omega \} \right] = \bigcup \{ \mathcal{P} : \mathcal{P} \text{ a } \lambda\text{-system over } \Omega \}$$

Proof:
$\lambda(\mathcal{P}) \in \sigma(\mathcal{P})$

By ③, $\sigma$-fields are $\lambda$-systems, so RHS is an $\bigcup$ of a larger collection of sets.
$\sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P})$ We'll show $\lambda(\mathcal{P})$ is a $\pi$-system. Exercise ② then implies $\lambda(\mathcal{P})$ is a $\sigma$-field, so $\sigma(\mathcal{P}) = \bigcap \{ \mathcal{P} : \mathcal{P} \text{ a } \sigma\text{-field over } \Omega \}$
Remains to prove $\lambda(P)$ is a $\pi$-system

Proof

Must prove: $E \cap F \in \lambda(P)$ for all $E, F \in \lambda(P)$

- Say $E \in \lambda(P)$ is cooperative if $E \cap P \in \lambda(P)$ for all $P \in P$.
- Say $E \in \lambda(P)$ is helpful if $E \cap F \in \lambda(P)$ for all $F \in \lambda(P)$.

If we show all elements of $\lambda(P)$ are helpful then we are done.

Which sets are cooperative?

- $\Omega$ is cooperative $\Omega \cap P = P \cap \lambda(P)$ for all $P \in P$.
- If $E \in P$ then $E \cap P \in P$ for all $P \in P$ so $E$ is cooperative.

All sets of $P$ cooperative

- If $E, F$ coop. then $\forall P \in P$, $E \cap P \in \lambda(P)$ and $F \cap P \in \lambda(P)$.
- If also $E \cap F$ then $E \cap P \cap F \cap P$, so $(F \cap E) \cap P = (F \cap P) \setminus (E \cap P) \in \lambda(P)$.

If $E, F$ coop. and $E \cap F$ then $F \setminus E$ coop.
If \((E_n, n \geq 1)\) increasing, \(E_n \uparrow E_\infty\), all \(E_n\) cooperative, then for all \(P \in \mathcal{P}\), \(E_n \cap P \in \lambda(P)\), and \(E_n \cap P \uparrow E_\infty \cap P\) so \(E_\infty \in \mathcal{P}\).

If \(E_n \uparrow E_\infty\), all \(E_n\) coop. then \(E_\infty\) coop

We have showed that \(\{\text{cooperative sets}\}\) is a \(\lambda\)-system containing \(P\), so all sets in \(\lambda(P)\) cooperative.

Which sets are helpful?

- \(\Omega\) is helpful
  \[\Omega \cap F = F \cap \lambda(P)\] for all \(F \in \lambda(P)\).
- If \(E \in \mathcal{P}\) then for all \(E \in \lambda(P)\), \(E\) cooperative so \(E \cap F \in \lambda(P)\); so \(E\) helpful.

All sets of \(\mathcal{P}\) are helpful

- If \(E, F\) helpful then \(\forall G \in \lambda(P)\), \(E \cap G \in \lambda(P)\) and \(F \cap G \in \lambda(P)\)
  \(\mathcal{P}\) a \(\lambda\)-system

If also \(E \subset F\) then \(E \cap G \subset F \cap G\), so \((F \setminus E) \cap G = (F \cap G) \setminus (E \cap G) \in \lambda(P)\)

If \(E, F\) helpful and \(E \subset F\) then \(F \setminus E\) helpful

Likewise, if \((E_n, n \geq 1)\) are helpful and \(E_n \uparrow E_\infty\) then \(E_\infty\) helpful.

So \(\{\text{helpful sets}\}\) is a \(\lambda\)-system containing \(P\); so all sets in \(\lambda(P)\) are helpful. So we are done.
Dynkin's Theorem: Let \((\Omega, \mathcal{F})\) be a set with a \(\sigma\)-field on it, and let \(PC\) be a \(\pi\)-system with \(\sigma(\mathcal{P})=\mathcal{F}\). If \(\mu_1, \mu_2\) are measures on \(\mathcal{F}\) and \(\mu_1(E)=\mu_2(E)\) for all \(E \in \mathcal{P}\) then \(\mu_1 \equiv \mu_2\).

Proof of Dynkin's Thm

Let \(\Lambda = \{F \in \mathcal{F} : \mu_1(F) = \mu_2(F)\}\). Then

- \(PC \Lambda\) by def.
- If \(E, F \in \Lambda, E \subseteq F\) then \(\mu_1(F \setminus E) = \mu_1(F) - \mu_1(E) = \mu_2(F) - \mu_2(E) = \mu_2(F \setminus E)\)

so \(\mu_1(F \setminus E) = \mu_2(F \setminus E)\). If \(E, F \in \Lambda, E \subseteq F\) then \(F \setminus E \in \Lambda\)

- If \(E_n \in \Lambda, n \geq 1\) and \(E_n \uparrow E_\infty\) then \(\lim_{n \to \infty} \mu_1(E_n) = \lim_{n \to \infty} \mu_2(E_n) = \mu_2(E_\infty)\)

If \(E_n \in \Lambda, n \geq 1\) and \(E_n \uparrow E_\infty\) then \(E_\infty \in \Lambda\)

Thus \(\Lambda\) is a \(\pi\)-system containing \(\mathcal{P}\), so \(\Lambda \supseteq \sigma(\mathcal{P})\), so \(\mu_1 \equiv \mu_2\).
Four notes about last class

Examples of \(\pi\)-systems \(\{\text{Open sets}\}; \{\text{Intervals}\}; \{\text{Boxes}\}\):

1. In hypothesis of Dynkin's Theorem we should assume that \(\Omega \in \mathcal{P}\), or equivalently that \(\mu_1(\Omega) = \mu_2(\Omega)\).

2. Def: Given a measurable space \((\Omega, \mathcal{F})\), a measure \(\mu\) on \(\mathcal{F}\) is \(\sigma\)-finite if there exist sets \((\Omega_n, n \geq 1)\) in \(\mathcal{F}\) s.t. \(\Omega_n \uparrow \Omega\) and \(\mu(\Omega_n) < \infty\) for all \(n\).

\(\sigma\)-finiteness should also appear in the hypothesis of Dynkin's Theorem.

In that case, if \(\mu_1(E) = \mu_2(E)\) and \(\mu_1(F) = \mu_2(F)\), and \(E \subseteq F\), then

\[
\mu_1(F \setminus E) = \lim_{n \to \infty} \mu_1((F \setminus E) \cap \Omega_n) = \lim_{n \to \infty} (\mu_1(F \cap \Omega_n) - \mu_1(E \cap \Omega_n))
\]

by the assumption.

\[
\mu_2(F \setminus E) = \lim_{n \to \infty} \mu_2((F \setminus E) \cap \Omega_n) = \lim_{n \to \infty} (\mu_2(F \cap \Omega_n) - \mu_2(E \cap \Omega_n))
\]

by the additivity of \(\mu_2\).

3. Recall: \(B(\mathbb{R}) = \sigma(\{U \subseteq \mathbb{R}: U \text{ open}\}) = \sigma(A(\mathbb{R}))\) is called the Borel sets of \(\mathbb{R}\).

Likewise \(B(\mathbb{R}^d) = \sigma(\{U \subseteq \mathbb{R}^d: U \text{ open}\})\).
**Key example:** Cumulative distribution functions / Stieltjes functions.

**Def:** A Stieltjes function is a function \( f : \mathbb{R} \to \mathbb{R} \) which is non-decreasing and right-continuous.

It is a CDF if \( F(-\infty) := \lim_{x \to -\infty} F(x) = 0 \) and \( F(\infty) := \lim_{x \to \infty} F(x) = 1 \).

**Prop:** Let \( F \) be a distribution function, and let \( A = \mathcal{A}(\mathbb{R}) := \{ (a_i, b_i] \cup \ldots \cup (a_k, b_k] \mid k \in \mathbb{N} \} \).

Define \( M = \mathcal{M}_F \) by \( M(\bigcup_{i=1}^{k} (a_i, b_i]) = \sum_{i=1}^{k} F(b_i) - F(a_i) \). \(-\infty < a_i \leq b_i \leq \ldots \leq a_k \leq b_k < \infty \).

Then \( M \) is a pre-measure on ring \( A \).

**Proof:** We prove this for the special case that \( F(x) = x \) (i.e. Lebesgue measure).

General case: same proof, more notation (in notes!).

**Step 1:** The definition makes sense.

Suppose \( \bigcup_{i=1}^{k} (a_i, b_i] = \bigcup_{i=1}^{m} (c_i, d_i] \), where both sides are disjoint unions.

Then for \( i \in [n], j \in [m] \), let \( S_{ij} = (a_i, b_i] \cap (c_j, d_j] \). If \( S_{ij} \neq \emptyset \) write \( S_{ij} = (l_{ij}, r_{ij}] \).

Then \( \sum_{i=1}^{k} (b_i - a_i) = \sum_{i=1}^{k} \sum_{j=1}^{m} (r_{ij} - l_{ij}) = \sum_{j=1}^{m} (d_j - c_j) \).

So def. makes sense.
Step 2 \( \mu \) is additive.

If \[ \bigcup_{i=1}^{n} (a_i, b_i) \cap \bigcup_{i=1}^{n} (c_i, d_i) = \emptyset \]

then

\[
\mu \left( \bigcup_{i=1}^{n} (a_i, b_i) \cup \bigcup_{i=1}^{n} (c_i, d_i) \right) = \sum_{i=1}^{n} (b_i - a_i) + \sum_{i=1}^{n} (d_i - c_i) = \mu \left( \bigcup_{i=1}^{n} (a_i, b_i) \right) + \mu \left( \bigcup_{i=1}^{n} (c_i, d_i) \right)
\]

So \( \mu \) is additive.

Step 3: \( \mu \) is a pre-measure.

We must show: if \( L = \bigcup_{i=1}^{n} (a_i, b_i) = \bigcup_{i=1}^{n} (c_i, d_i) \), where both are disjoint unions,
then

\[\mu(L) = \sum_{i=1}^{n} (b_i - a_i) = \sum_{i=1}^{n} (d_i - c_i)\]

For all \( m \), \( L \supseteq \bigcup_{i=1}^{m} (c_i, d_i) \), so

\[\mu(L) \geq \sum_{i=1}^{m} (d_i - c_i) = \sum_{i=1}^{m} (d_i - c_i)\]

Thus

\[\mu(L) \geq \sum_{i=1}^{m} (d_i - c_i)\]

For \( m \geq 0 \), write \( \Delta_m = L \setminus \bigcup_{i=1}^{m} (c_i, d_i) \).

We'll show the two sides weren't equal after all.
Then $\Delta_m = \bigcup_{i=1}^{m}(a_i, b_i)$, $\bigcap_{i=1}^{m}(c_i, d_i) \in A$ and $\Delta_m > \Delta_{m+1}$.

with $\Delta_m \downarrow 0$ as $m \to \infty$.

Also, $\mu(\Delta_m)$

$$= \mu\left(L \setminus \bigcup_{i=1}^{m}(c_i, d_i)\right)$$

$$= \mu(L) - \mu\left(\bigcup_{i=1}^{m}(c_i, d_i)\right) = \mu(L) - \sum_{i=1}^{m} (d_i - c_i) \geq 2\varepsilon.$$ 

Choose $D_m \in A$ with $\overline{D}_m \subset \Delta_m$ s.t. $\mu(\Delta_m \setminus D_m) \leq \frac{\varepsilon}{2^m}$.

For $x \in \Delta_m$, if $\exists i$ s.t. $x \notin D_i$ then $x \in \Delta_i \setminus D_i$

Note that $\Delta_m = \bigcap_{i=1}^{m} D_i \cup \bigcup_{i=1}^{m} (\Delta_m \setminus D_i) \subseteq \bigcap_{i=1}^{m} D_i \cup \bigcup_{i=1}^{m} (\Delta_i \setminus D_i)$, so

monotonicity

$$\mu(\Delta_m) \leq \mu\left(\bigcap_{i=1}^{m} D_i \cup \bigcup_{i=1}^{m} (\Delta_i \setminus D_i)\right)$$

subadditivity

$$\leq \mu\left(\bigcap_{i=1}^{m} D_i\right) + \sum_{i=1}^{m} \mu(\Delta_i \setminus D_i) \leq \sum_{i=1}^{m} \frac{\varepsilon}{2^i} < \varepsilon.$$
Also, \( \mu(\Delta_m) \geq 2\varepsilon \), so \( \mu(\bigcap_i \overline{D_i}) \geq \varepsilon \).

Thus \( \bigcap_i \overline{D_i} \neq \emptyset \) for all \( m \), so \( \bigcap_i \overline{D_i} \neq \emptyset \). But \( \bigcap_i \overline{D_i} \in \bigcap_i \Delta_m = \emptyset \), a contradiction. \( \square \)

**Theorem:** Let \( F \) be a Stieltjes function. Then there exists a unique measure \( \mu \) on \( \sigma(\mathcal{AC}_\mathbb{R}) \) s.t. \( \mu([a,b]) = F(b) - F(a) \) for all \( -\infty < a < b < \infty \).

**Proof:**

Existence

By the proposition, there exists a pre-measure \( \mu \) on \( \mathcal{AC}_\mathbb{R} \) with this property.

By CXT, \( \mu \) extends to a measure on \( L(\mathcal{AC}_\mathbb{R}) \supseteq \sigma(\mathcal{AC}_\mathbb{R}) \),

Uniqueness

Suppose \( \mu_1, \mu_2 \) are as in the Theorem statement.

Let \( P = \{ (a,b) : -\infty < a \leq b < \infty \} \).

Then \( P \) is a \( \pi \)-system with \( \sigma(P) = \sigma(\mathcal{AC}_\mathbb{R}) \), so by Dynkin's theorem \( \mu_1 = \mu_2 \).

**Exercise** If \( \mu \) is a \( \sigma \)-finite measure on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) then there exists a Stieltjes \( F \) s.t. \( \mu_F = \mu \).