Math 587 – Fall 2019 – Assignment 6

Assigned on November 20, 2019.
Due on December 3, 2019 at 2 PM

1. [Convex implies continuous]
   Let $f : \mathbb{R} \to \mathbb{R}$ be convex. Prove that f is continuous.

2. [L_p convergence almost surely determines the limit.]
   Prove that for any $p > 0$ and any random variables $(X_n, n \geq 1)$ and X, Y in $L_p(\Omega, \mathcal{F}, \mathbf{P})$, if $X_n \xrightarrow{L_p} X$ and $X_n \xrightarrow{L_p} Y$ then $X \overset{\text{a.s.}}{=} Y$.

3. [Test functions that are indicators determine a random variable.]
   Let $X, Y \in L_1(\Omega, \mathcal{F}, \mathbf{P})$. Suppose that $\mathbf{E}\left[X\mathbf{1}_{[E]}\right] = \mathbf{E}\left[Y\mathbf{1}_{[E]}\right]$ for all $E \in \mathcal{F}$. Prove that $X \overset{\text{a.s.}}{=} Y$.

4. [Adding an independent conditioning changes nothing.]
   Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $X \in L_1(\Omega, \mathcal{F}, \mathbf{P})$. Let $\mathcal{G}, \mathcal{H}$ be two sub-σ-fields of $\mathcal{F}$. Suppose that $\sigma(X, \mathcal{G}) := \sigma(\sigma(X) \cup \mathcal{G})$ is independent of $\mathcal{H}$. Let Z be a version of $\mathbf{E}\{X \mid \mathcal{G}\}$.

   (i) Show that for all $A \in \mathcal{G}$ and $B \in \mathcal{H}$,
   $$\mathbf{E}\left[X\mathbf{1}_{[A\cap B]}\right] = \mathbf{E}\left[Z\mathbf{1}_{[A\cap B]}\right].$$
   (ii) Let $\mathcal{S} = \{E \in \sigma(\mathcal{G}, \mathcal{H}) : \mathbf{E}\left[X\mathbf{1}_{[E]}\right] = \mathbf{E}\left[Z\mathbf{1}_{[E]}\right]\}$. Show that $\mathcal{S}$ is a λ-system.

   (iii) Show that $\mathbf{E}\{X \mid \sigma(\mathcal{G}, \mathcal{H})\} \overset{\text{a.s.}}{=} Z$.

5. [Conditional Chebyshev Inequality]
   Let $X \in L_2(\Omega, \mathcal{F}, \mathbf{P})$. Prove that for all $t > 0$,
   $$\mathbf{P}\{|X| \geq t \mid \mathcal{G}\} \leq \frac{1}{t^2}\mathbf{E}\{X^2 \mid \mathcal{G}\}.$$
   (Here you should understand $\mathbf{P}\{|X| \geq t \mid \mathcal{G}\} := \mathbf{E}\{\mathbf{1}_{[|X|\geq t]} \mid \mathcal{G}\}$.)

6. [When the tower falls]
   Construct an example of a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a random variable $X \in L_1(\Omega, \mathcal{F}, \mathbf{P})$, and sub-$\sigma$-fields $\mathcal{G}, \mathcal{H}$ of $\mathcal{F}$ such that
   $$\mathbf{E}\{\mathbf{E}\{X \mid \mathcal{G}\} \mid \mathcal{H}\} \neq \mathbf{E}\{\mathbf{E}\{X \mid \mathcal{H}\} \mid \mathcal{G}\}.$$

7. [Projection is norm-decreasing]

   Let $X \in L_2(\Omega, \mathcal{F}, \mathbf{P})$ and $Y \in L_2(\Omega, \mathcal{G}, \mathbf{P})$, where $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$. Suppose that $Y \overset{\text{a.s.}}{=} \mathbf{E}\{X \mid \mathcal{G}\}$ and that $\|X\|_2 = \|Y\|_2$. Prove that in fact $X \overset{\text{a.s.}}{=} Y$.

8. [Filtering a single random variable]
   Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbf{P})$ be a filtered probability space and let $X \in L_1(\Omega, \mathcal{F}, \mathbf{P})$. Write $X_n \overset{\text{a.s.}}{=} \mathbf{E}\{X \mid \mathcal{F}_n\}$. Show that $(X_n, n \geq 0)$ is a martingale relative to the filtration $(\mathcal{F}_n, n \geq 0)$.