Math 587 – Fall 2019 – Assignment 5

Assigned on November 4, 2019.
Due on November 15 at 2 PM

1. [Lacunary SLLN implies Monotone SLLN] Let $(s_n, n \geq 0)$ be a non-decreasing sequence with $s_0 = 0$. Fix $\mu > 0$, $\epsilon \in (0, 1/3)$, and define a sequence by $n_k = \lceil (1+\epsilon)^k \rceil$.

   (a) Show that for all n sufficiently large (i.e. $n \geq n_0(\epsilon)$), if $s_n \geq \mu n(1 + 3\epsilon)$ then letting k be such that $n_{k-1} < n \leq n_k$, we have $s_{n_k} \geq \mu n_k(1 + \epsilon)$.

   (b) Show that for all n sufficiently large, if $s_n \leq \mu n(1 - 3\epsilon)$ then letting k be such that $n_{k-1} < n \leq n_k$, we have $s_{n_{k-1}} \leq \mu n_{k-1}(1 - \epsilon)$.

   (c) Conclude that if $\limsup_n |s_n - \mu n| > 3\epsilon\mu n$ then $\limsup_k |s_{n_k} - \mu n_k| > \epsilon\mu n_k$.

2. [Convolution and densities]

   (a) Show that if X, Y are independent and f is a probability density function (pdf) for X and g is a pdf for Y then $f * g$ is a pdf for $X + Y$.

   (b) Show that if X and Y are independent Gaussian random variables (possibly with different means and variances) then $X + Y$ is again a Gaussian.

   (c) Show that if X and Y are independent Poisson random variables (possibly with different means) then $X + Y$ is again a Poisson.

3. [Convolution, Cauchy random variables, and Convergence] For $u > 0$ let $c_u : \mathbb{R} \to [0, \infty)$ be defined by

$$c_u(x) = \frac{1}{\pi} \frac{u}{u^2 + x^2} .$$

   (a) Prove that $\int_{\mathbb{R}} c_u(x) \mathrm{d}x = 1$, so c_u is a probability density function (the *Cauchy*(u) density.

   (b) Show that for $u, v > 0$ we have $c_u * c_v = c_{u+v}$.

   For the remaining parts of this question, fix $u > 0$ and let $(X_i, i \geq 1)$ be independent Cauchy(u) random variables (i.e. all have density c_u). Also, write $S_n = \sum_{i=1}^n X_i$ for $n \geq 1$.

   (c) Show that for $u > 0$, if $X_1, \ldots, X_n$ are independent and all have density c_u, then S_n/n has density c_u.

   (d) Show that $\mathbf{P}\{\lim_{n\to\infty} S_n/n \text{ exists and is finite}\} = 0$.

   (e) Show that for $x > 0$,

$$\mathbf{P}\left\{ n^{-1} \max_{1 \leq k \leq n} X_k \leq x \right\} \to e^{-u/\pi x}$$

   as $n \to \infty$.

4. [Maximal inequalities for sums of independent random variables] In this question $(X_i, i \geq 1)$ are mutually independent with $\mathbf{E}[X_i] = 0$ for all i and with $\mathbf{Var}\{X_i\} < \infty$ for all i. We write $S_n = \sum_{i=1}^n X_i$ for $n \geq 1$.

   (a) Show that for all $1 \leq k \leq n$, for any event $A \in \sigma(X_1, \ldots, X_k)$, the random variables $S_k \mathbf{1}_{[A]}$ and $(S_n - S_k)$ are independent. Deduce that

   $$\mathbf{E}\left[S_k(S_n - S_k)\mathbf{1}_{[A]}\right] = 0.$$

   (b) Fix $t > 0$ and let $\tau = \min\{i \geq 1 : |S_i| \geq t\}$. Show that for all $1 \leq k \leq n$,

   $$\mathbf{E}\left[S_n^2 \mathbf{1}_{[\tau=k]}\right] \geq t^2 \mathbf{P}\{\tau = k\}.$$

   (Hint: first show that $\{\tau = k\} \in \sigma(X_1, \ldots, X_k)$.)

   (c) Prove that for all $t > 0$,

   $$\mathbf{P}\left\{\max_{1\leq k\leq n} |S_k| \geq t\right\} \leq \frac{\mathbf{Var}\{S_n\}}{t^2}.$$

   (Hint: $\max_{1\leq k\leq n|} |S_k|$ if and only if $\tau = k$ for some $1 \leq k \leq n$.)

5. [Randomly stopped sums] Let $(X_i, i \geq 1)$ be independent and identically distributed with $\mathbf{E}|X_1| < \infty$. Let N be a positive integer random variable which is independent of $(X_i, i \geq 1)$.

   (a) Show that if $\mathbf{E}N < \infty$ then

   $$\mathbf{E}[X_1 + \ldots + X_N] = \mathbf{E}N \cdot \mathbf{E}X_1.$$

   (b) Find an example which shows that the conclusion of (a) need not hold if N is allowed to depend on the random variables $(X_i, i \geq 1)$.

   (c) Write F for the cumulative distribution function of the X_i and

   $$F^{(n)} = F * F * \ldots * F$$

   for the n-fold convolution of F. Show that the CDF of S_n satisfies

   $$F_{S_N}(x) = \sum_{n\geq 1} \mathbf{P}\{N = n\} F^{(n)}(x).$$