Math 587 – Fall 2019 – Assignment 4

Assigned on October 23, 2019.
Due on November 4, 2019 at 2 PM

1. [The Gaussian integral]

   (a) Use change of variables and Fubini's theorem to prove that $(\int_{\mathbb{R}} e^{-x^2} \mathrm{d}x)^2 = \pi$. [You've perhaps seen this before and know how the proof goes. If not: look for an integral over $\mathbb{R}^2$, and consider a switch to polar coordinates. You can easily look this up but try to prove it on your own.]

   (b) For $\alpha \in \mathbb{R}$ and $\sigma > 0$, the Normal(α, σ^2) density is given by

   $$\varphi_{\alpha,\sigma^2}(x) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\alpha)^2/(2\sigma^2)} .$$

   Show that if X and Y are independent normals with densities $\varphi_{\alpha,\sigma^2}(x)$ and φ_{β,τ^2} respectively, then $X+Y$ has density $\varphi_{\alpha+\beta,\sigma^2+\tau^2}$; in particular $X+Y$ is again a normal random variable.

2. [Some properties of product spaces] Let $(M, \mathcal{M})$ and $(N, \mathcal{N})$ be measurable spaces.

   (a) Show that if $E \in \mathcal{M} \otimes \mathcal{N}$ then for all $a \in M$, the section $E_a := \{b \in N : (a,b) \in E\} \in \mathcal{N}$.

   (b) Show that if $\emptyset \neq A \subset M$, $\emptyset \neq B \subset N$, and $A \times B \in \mathcal{M} \otimes \mathcal{N}$ then $A \in \mathcal{M}$ and $B \in \mathcal{N}$.

   (c) Suppose that $M = N$ is an uncountable set and that

   $$\mathcal{M} = \mathcal{N} = \{A \subset M : \text{ either } A \text{ or } M \setminus A \text{ is countable}\} .$$

   Show that the diagonal $D = \{(a,a), a \in M\} \subset M \times N$ is not an element of $\mathcal{M} \otimes \mathcal{N}$.

3. [Cavalieri's principle] Let $(M, \mathcal{M})$ and $(N, \mathcal{N})$ be finite measure spaces and let $E, F \in \mathcal{M} \otimes \mathcal{N}$. Suppose that $\nu(E_a) = \nu(F_a)$ for all $a \in M$, where E_a and F_a are sections as in question 2 (a). Prove that $\mu \otimes \nu(E) = \mu \otimes \nu(F)$.

4. [A Fubini warning] [For the next question, we begin by noting that if μ is counting measure on $(\mathbb{N}, 2^{\mathbb{N}})$ and $f : \mathbb{N} \to \mathbb{R}$ then $f \in L_1(\mu)$ if and only if $\sum_{n \in \mathbb{N}} |f(n)| < \infty$, and if $f \in L_1(\mu)$ or $f \geq 0$ then $\int f \mathrm{d}\mu = \sum_{n \in \mathbb{N}} f(n)$. You should be able to prove this but don't need to hand it in for the assignment.]

   Let μ and ν be counting measure on $(\mathbb{N}, 2^{\mathbb{N}})$, and define $f : \mathbb{N}^2 \to \mathbb{R}$ by

   $$f(i,j) = \begin{cases} 2 - \frac{1}{2^i} & \text{if } i = j \\ \frac{1}{2^i} - 2 & \text{if } i = j+1 \\ 0 & \text{otherwise} \end{cases}$$

Prove that the iterated integrals $\int_{\mathbb{N}} \int_{\mathbb{N}} f(i,j)\nu(\mathrm{d}j)\mu(\mathrm{d}i)$ and $\int_{\mathbb{N}} \int_{\mathbb{N}} f(i,j)\mu(\mathrm{d}i)\nu(\mathrm{d}j)$ both exist but are not equal. [Make sure you understand why this doesn't contradict Fubini's theorem.]

5. [A Tonelli warning] Let $f : [-1,1]^2 \to \mathbb{R}$ be given by $f(x,y) = xy/(x^2+y^2)^2$. (You may take $f(0,0) = 0$ if you like.)

   Show that the iterated integral $\int_{[-1,1]} \int_{[-1,1]} f(x,y)\mathrm{d}x\mathrm{d}y$ exists but that f is not in $L_1(\mathrm{Leb}_{[-1,1]^2})$.

6. [Some Fubini identities for random variables] Let X be a random variable and let F be its CDF.

   (a) Prove that for all $c > 0$,
   $$\int_{\mathbb{R}} (F(x+c) - F(x))\mathrm{d}x = c.$$

   (b) Prove that if $\mathbf{E}|X| < \infty$ then
   $$\mathbf{E}X = \int_{[0,\infty)} \mathbf{P}\{X > t\}\,\mathrm{d}t - \int_{(-\infty,0]} \mathbf{P}\{X < t\}\,\mathrm{d}t\,.$$

   (c) Let Y be another random variable defined on the same space as X and with $\mathbf{P}\{Y > X\} = 1$. Show that $\mathbf{E}[Y - X] = \int_{\mathbb{R}} \mathbf{P}\{t \in (X,Y)\}\,\mathrm{d}t$.

7. [¡¡The product formula does not imply independence!!] Let X be a random variable. Find measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that, with $Y = f(X)$, then $\mathbf{E}|Y| < \infty$, $\mathbf{E}|XY| < \infty$, and $\mathbf{E}[XY] = \mathbf{E}[X]\,\mathbf{E}[Y]$, when X has any of the following distributions. Justify your answers.

   (i) X is Normal$(0,1)$.

   (ii) X is Exponential(1).

   (iii) X is Poisson(1).

8. [A strengthening of Chebyshev's inequality] Let $(X_i, i \geq 1)$ be independent identically distributed random variables with $\mathbf{E}\left[|X_i|^2\right] < \infty$. Show that for any $\epsilon > 0$,
   $$\lim_{n\to\infty} n\mathbf{P}\left\{|X_1| \geq \epsilon\sqrt{n}\right\} = 0\,,$$
   and that $n^{-1/2}\max_{1\leq k\leq n} X_k \to 0$ in probability.