Math 587 – Fall 2019 – Assignment 3

Assigned on October 3, 2019.
Due on October 14, 2019 at 2 PM

1. [Convergence in distribution] Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $(X_n, 1 \le n \le \infty)$ be real-valued random variables defined on Ω. For a set $S \subset \mathbb{R}$, the *boundary of* S is $\partial S := \{x \in \mathbb{R} : \forall c > 0, B(x,c) \cap S \neq \emptyset, B(x,c) \cap (\mathbb{R} \setminus S) \neq \emptyset\}$.

   (a) Show that if $S \in \mathcal{B}(\mathbb{R})$ then $\partial S \in \mathcal{B}(\mathbb{R})$.

   (b) Show that $X_n \xrightarrow{\mathrm{d}} X_\infty$ if and only if $\mathbf{P}\{X_n \in S\} \to \mathbf{P}\{X_\infty \in S\}$ for all Borel sets S with $\mathbf{P}\{X_\infty \in \partial S = 0\}$.

   (c) Show that $X_n \xrightarrow{\mathrm{d}} X_\infty$ if and only if for any subsequence $(n_k, k \ge 1)$ of $\mathbb{N}$ there is a subsubsequence $(m_k, k \ge 1)$ of $(n_k, k \ge 1)$ such that$X_{m_k} \xrightarrow{\mathrm{d}} X_\infty$.

2. [Stochastic boundedness and convergence in distribution.] Let $X = (X_i, i \in I)$ be a collection of real random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We say that X is *stochastically bounded* if for all $\epsilon > 0$ there is a compact set $K \subset \mathbb{R}$ such that $\inf_{i \in I} \mathbf{P}\{X_i \in K\} > 1 - \epsilon$.

   Show that if $(X_n, n \ge 1)$ is a stochastically bounded collection then there exists a subsequence $(n_k, k \ge 1)$ such that $(X_{n_k}, k \ge 1)$ converges in distribution.

3. Integrable functions are almost continuous Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function with $f \in L_1$, i.e. with $\int_{\mathbb{R}} |f(x)| \mathrm{d}x < \infty$. Show that for all $\epsilon > 0$ there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ with $g \in L_1$ such that $\int |f(x) - g(x)| \mathrm{d}x < \epsilon$.

4. Cosine cancels integrable functions. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function with $f \in L_1$. Prove that $\int_R f(x) \cos(ax) \mathrm{d}x \to 0$ as $a \to \infty$. (Hint: first assume f is simple.)

5. [Uniform integrability] Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\mu(\Omega) < \infty$, and let $F = (f_i, i \in I)$ be $(\mathcal{F}/\mathcal{B}(\mathbb{R}))$-measurable functions. We say the collection F is *uniformly integrable* if

$$\lim_{M \to \infty} \sup_{i \in I} \int_\Omega |f_i| \mathbf{1}_{[|f_i| > M]} \mathrm{d}\mu = 0.$$

   (a) Prove that for all $i \in I$, if f_i is integrable then $\lim_{M \to \infty} \int_\Omega |f_i| \mathbf{1}_{[|f_i| > M]} \mathrm{d}\mu = 0$.

   (b) Prove that for all $i \in I$, if f_i is integrable then for any $\epsilon > 0$ there is $\delta = \delta(\epsilon, f_i) > 0$ such that for all $B \in \mathcal{F}$ with $\mu(B) < \delta$,

$$\int_B f_i \mathrm{d}\mu := \int f_i \mathbf{1}_{[B]} \mathrm{d}\mu < \epsilon.$$

   (c) Prove that F is uniformly integrable if and only if (i) $\sup_{i \in I} \int |f_i| \mathrm{d}\mu < \infty$ and (ii) for all $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that for all $B \in \mathcal{F}$ with $\mu(B) < \delta$,

$$\sup_{i \in I} \int_B |f_i| \mathrm{d}\mu < \epsilon.$$

   (d) Show by example that neither (i) nor (ii) in part (c) implies the other.