Math 587 – Fall 2019 – Assignment 2

Assigned on September 18 2019.
Due on ~~September 27~~ (Climate march!) September 30, 2019 at 2 PM
(by email or in my office)

This assignment is mostly about independence, and a little about measurable sets and measurable maps.

1. [Borel sets]

   (a) Show that $\mathcal{B}(\mathbb{R}^d) = \sigma(\{B(x,q), x \in \mathbb{Q}^d, q \in \mathbb{Q}\})$. Here $B(x,q) := \{y \in \mathbb{R}^d : |y-x| < q\}$. In other words, the Borel sets in $\mathbb{R}^d$ are generated by open balls with rational centre and rational radius, so in particular $\mathcal{B}(\mathbb{R}^d)$ is *countably generated*.

   (b) Show that $\mathcal{B}(\mathbb{R}) = \sigma(\{(a,b], -\infty < a \le b < \infty\} = \sigma(\{(-\infty,b], -\infty < b < \infty\})$, and that

   $$\mathcal{B}(\mathbb{R}^d) = \sigma\left((a_1,b_1] \times \ldots \times (a_d,b_d] : -\infty < a_i \le b_i < \infty \text{ for } 1 \le i \le d,\right).$$

2. [Stieltjes functions] Prove that if μ is a σ-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ then there exists a Stieltjes function F such that $\mu_F = \mu$.

3. [Independent events and σ-fields] Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space.

   (a) Prove that events $(E_i, i \ge 1)$ in $\mathcal{F}$ are mutually independent if and only if the σ-fields $\{\{\emptyset, E_i, E_i^c, \Omega\}, i \ge 1\}$ are independent.

   (b) Prove that if events $(E_i, i \in I)$ in $\mathcal{F}$ are mutually independent and $(I_n, n \ge 1)$ is any partition of I, then the σ-fields $(\sigma(\{E_i, i \in I_n\}), n \ge 1)$ are independent.

4. [k-wise independence.]

   In this exercise we say that an event E in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is *non-trivial* if $\mathbf{P}\{E\} \in (0,1)$.

   (a) Let $k \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $(E_1, \ldots, E_k)$ be nontrivial, independent events in $\mathcal{F}$. Prove that $|\Omega| \ge 2^k$.

   (b) Construct a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with $|\Omega| = 2^k$ and nontrivial events $(E_1, \ldots, E_k)$, such that for any $1 \le i \le n$, the events $(E_j, j \in [k] \setminus \{i\})$ are mutually independent, but $(E_1, \ldots, E_k)$ are not mutually independent.

5. [Finite-range dependence and the second Borel-Cantelli lemma] Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $(E_n, n \in \mathbb{N})$ be events in $\mathcal{F}$. Say that the collection $(E_n, n \in \mathbb{N})$ has a *finite range of dependency* if there exists $r > 0$ such that the following holds: If $S \subset \mathbb{N}$ is such that any distinct elements $i, j \in S$ satisfy $|i-j| > r$, then the random variables $(E_n, n \in S)$ are mutually independent.

   Prove that if $(E_n, n \ge 1)$ has a finite range of dependency and $\sum_{n \ge 1} \mathbf{P}\{E_n\} = \infty$ then $\mathbf{P}\{E_n \text{ i.o.}\} = 1$.

6. [Basic operations preserve measurability of maps] Let $(\Omega, \mathcal{F})$ be a measurable space and let X, Y, and $(X_n, n \geq 1)$ be $(\mathcal{F}/\mathcal{B}(\mathbb{R}))$-measurable maps from Ω to $\mathbb{R}$.

   (a) Prove that $\mathbf{1}_{[X\geq 0]}, X+Y, XY, (X/Y)\mathbf{1}_{[Y\neq 0]}$ are all $(\mathcal{F}/\mathcal{B}(\mathbb{R}))$-measurable.

   (b) Prove that $\sup_{n\geq 1} X_n, \inf_{n\geq 1} X_n, \limsup_{n\geq 1} X_n$ and $\liminf_{n\geq 1} X_n$ are all $(\mathcal{F}/\mathcal{B}(\mathbb{R}^*))$-measurable.

   (c) Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is $(\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}))$-measurable then $f(X_1, \ldots, X_n)$ is $(\mathcal{F}/\mathcal{B}(\mathbb{R}))$-measurable.

7. [An ultra-weak law of large numbers] Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}(\mathbb{R})|_{[0,1]}$, and $\mathbf{P}$ be Lebesgue measure on $[0,1]$. Recall from class the events $(A_k, k \geq 1)$ given by

$$A_k = \bigcup_{0\leq i<2^k, i \text{ even}} \left(\frac{i}{2^k}, \frac{i+1}{2^k}\right] .$$

   (a) Show that $(A_k, k \geq 1)$ are mutually independent.

   (b) Fix $M \geq 3$ odd, and for $k \geq 1$ let

$$E_k^M = \left\{x \in [0,1] : x \text{ is in at least half of the sets } \{A_{M^{k-1}+1}, \ldots, A_{M^k}\}\right\} .$$

   Prove that $E_k^M \in \mathcal{F}$ and that $\mathbf{P}\left\{E_k^M\right\} = 1/2$.

   (c) Prove that the events $(E_k^M, k \geq 1)$ are mutually independent.

   (d) Prove that if $x \in \{E_k^M \text{ i.o.}\} = \bigcap_{\ell\geq 1}\bigcup_{k\geq \ell} E_\ell^M$ then

$$\limsup_{n\to\infty} \frac{\#\{1 \leq i \leq n : x \in A_i\}}{n} \geq \frac{M-1}{2M}.$$

   (e) Write

$$E^+ = \left\{x : \limsup_{n\to\infty} \frac{\#\{1 \leq i \leq n : x \in A_i\}}{n} \geq \frac{1}{2}\right\}$$

   and

$$E^- = \left\{x : \limsup_{n\to\infty} \frac{\#\{1 \leq i \leq n : x \in A_i\}}{n} \leq \frac{1}{2}\right\} .$$

   Prove that $E^+ \in \mathcal{F}$ and $E^- \in \mathcal{F}$ and (perhaps using Borel-Cantelli 2) that

$$\mathbf{P}\left\{E^+ \cap E^-\right\} = 1 .$$

8. [Push-forwards and pull-backs] Let $(\Omega, \mathcal{F})$ be a measurable space. Given a set Y and a function $f : \Omega \to Y$, the *push-forward* of $\mathcal{F}$ to Y is the set $f^*(\mathcal{F}) = \{B \subset Y : f^{-1}(B) \in \mathcal{F}\}$.

   (a) Show that $f^*(\mathcal{F})$ is a σ-field.

   (c) If X, Y are two topological spaces and $f : X \to Y$ is continuous, prove that $f^{-1}(B)$ is Borel for every Borel set B in Y.