Math 587 – Fall 2019 – Assignment 1

Assigned on September 5, 2019.
Due on September 16, 2019 at 2 PM (by email or in my office)

This assignment about basic properties of rings, fields, measurable spaces, measure spaces and measures.

1. [Restrictions of measures.] Let $(\Omega, \mathcal{F})$ be a measurable space. Fix $E \subset \Omega$ and let

$$\mathcal{F}|_E := \{F \cap E : F \in \mathcal{F}\}.$$

Show that $\mathcal{F}|_E$ is a σ-field.

Proof. We prove that $\mathcal{F}|_E$ is a σ-field over E by verifying $\mathcal{F}|_E$ contains E and is closed under complements (relative to E) and countable disjoint unions. In what follows we write F^c to mean $\Omega \setminus F$.

First, $\Omega \in \mathcal{F}$ so $E = \Omega \cap E \in \mathcal{F}|_E$.

Next, if $G \in \mathcal{F}|_E$ then there is $F \in \mathcal{F}$ such that $G = F \cap E$. But then $E \setminus G = E \setminus F = E \cap F^c$, so $\mathcal{F}|_E$ is closed under complements.

Finally, if $(G_n, n \geq 1)$ are disjoint elements of $\mathcal{F}|_E$ then there are elements $(F_n, n \geq 1)$ of $\mathcal{F}$ such that $G_n = F_n \cap E$. But then

$$\bigcup_{n \geq 1} G_n = \bigcup_{n \geq 1} F_n \cap E = \left(\bigcup_{n \geq 1} F_n\right) \cap E.$$

Moreover, $\bigcup_{n \geq 1} F_n \in \mathcal{F}$ since $\mathcal{F}$ is a σ-fields; thus $\bigcup_{n \geq 1} G_n$. □

2. [Basic properties of measures]
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

   (a) Show that if $E, F \in \mathcal{F}$ and $E \subset F$ then $\mu(E) \leq \mu(F)$. (Note: prove this assuming only that μ is countably additive and that $\mu(\emptyset) = 0$.)

   (b) Show that if $(E_i, i \geq 1)$ is an increasing sequence of elements of $\mathcal{F}$, in that $E_i \subseteq E_j$ for $i \leq j$, then $\mu(\bigcup_{i \geq 1} E_i) = \lim_{i \to \infty} \mu(E_i)$.

   (c) Show that if $(E_i, i \geq 1)$ is a decreasing sequence of elements of $\mathcal{F}$ and $\mu(E_1) < \infty$ then then $\mu(\bigcap_{i \geq 1} E_i) = \lim_{i \to \infty} \mu(E_i)$.

Proof. (a) If $E, F \in \mathcal{F}$ and $E \subset F$ then since $\mathcal{F}$ is a σ-field, $F \setminus E \in \mathcal{F}$. Since μ is countably additive, it follows that

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$

(b) Let $F_i = E_i \setminus E_{i-1}$. Then the sets $(F_i, i \geq 1)$ are disjoint and $\bigcup_{i\geq 1} F_i = \bigcup_{i\geq 1} E_i$. It follows by the countable additivity of μ that

$$\begin{aligned}\mu(\bigcup_{i\geq 1} E_i) &= \mu(\bigcup_{i\geq 1} F_i) \\ &= \sum_{i\geq 1} \mu(F_i) \\ &= \lim_{i\to\infty} \sum_{j=1}^{i} \mu(F_j) \\ &= \lim_{i\to\infty} \mu(\bigcup_{j=1}^{i} F_j)\,,\end{aligned}$$

where in the last line we have again used (finite) additivity of μ. But $\bigcup_{j=1}^{i} F_j = E_i$, so the last expression is just $\lim_{i\to\infty} \mu(E_i)$.

(c) Write $E'_i = E_1 \setminus E_i$. Then the sets $(E'_i, i \geq 1)$ are increasing and their limit is $E_1 \setminus \bigcap_{i\geq 1} E_i$. It then follows from the result of part (b) applied to the sequence $(E'_i, i \geq 1)$

$$\mu(E_1 \setminus \bigcap_{i\geq 1} E_i) = \lim_{i\to\infty} \mu(E_1 \setminus E_i).$$

If $\mu(E_1)$ is finite, then by additivity this implies that

$$\mu(E_1) - \mu(\bigcap_{i\geq 1} E_i) = \lim_{i\to\infty} \mu(E_1) - \mu(E_i),$$

which gives the result. □

3. [Null sets]
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Say $N \in \mathcal{F}$ is a *null set* if $\mu(N) = 0$. Say that $(\Omega, \mathcal{F}, \mu)$ is *complete* if for any null set N, for all $M \subset N$ we have $M \in \mathcal{F}$. Write $\overline{\mathcal{F}} := \bigcap_{\{\mathcal{F}'\supset\mathcal{F}:(\Omega,\mathcal{F}',\mu)\text{ complete}\}} \mathcal{F}'$. Prove that

$$\overline{\mathcal{F}} = \{E \cup M : E \in \mathcal{F},\ \text{there exists a null set } N \in \mathcal{F} \text{ such that } M \subset N\}\,.$$

Proof. **Step 1.** Write

$$\mathcal{F}_1 = \{E \cup M : E \in \mathcal{F},\ \text{there exists a null set } N \in \mathcal{F} \text{ such that } M \subset N\},$$

and let $\mu_1 : \mathcal{F}_1 \to \mathbb{R}$ be defined by setting $\mu_1(E \cup M) = \mu(E)$ if $E \in \mathcal{F}$ and $M \subset N$ for some $N \in \mathcal{F}$ with $\mu(N) = 0$. Note that if $E \cup M = E' \cup M'$ are distinct representations of this form then

$$E' = (E \cup M) \setminus M' \supset E \setminus N$$

so $\mu(E') \geq \mu(E) - \mu(N) = \mu(E)$. By symmetry, it follows that $\mu(E) = \mu(E')$, so the definition of $\mu_1(E \cup M)$ makes sense (it does not depend on the representation of $E \cup M$).

Step 2. Let $(\Omega, \mathcal{F}', \mu')$ be any complete extension of $(\Omega, \mathcal{F}, \mu)$. This means that $(\Omega, \mathcal{F}', \mu')$ is complete, and $\mathcal{F} \subset \mathcal{F}'$, and $\mu'(E) = \mu(E)$ for all $E \in \mathcal{F}$.

Fix any null set $N \in \mathcal{F}$. Since $\mathcal{F} \subset \mathcal{F}'$, necessarily $N \in \mathcal{F}'$. Since $(\Omega, \mathcal{F}', \mu')$ is complete, it follows that $M \in \mathcal{F}'$ for all $M \subset N$. Since $\mathcal{F}'$ is a σ-field, it follows that $E \cup M \in \mathcal{F}'$ for all $E \in \mathcal{F}$. Thus $\mathcal{F}' \supset \mathcal{F}_1$.

Step 2. Fix $E' \in \mathcal{F}_1$. Then we may write $E' = E \cup M$ where M is a subset of $N \in \mathcal{F}$ with $\mu(N) = 0$. Thus, by monotonicity of measures

$$\mu(E) = \mu'(E) \leq \mu'(E') \leq \mu'(E) + \mu'(M) \leq \mu'(E) + \mu'(N) = \mu(E) + \mu(N) = \mu(E)\,.$$

In other words, $\mu'(E') = \mu(E) = \mu_1(E)$.

Step 3. We have now proved that if $(\Omega, \mathcal{F}', \mu')$ is any complete extension of $(\Omega, \mathcal{F}, \mu)$ then $\mathcal{F}' \supset \mathcal{F}_1$ and $\mu'(E') = \mu_1(E')$ for all $E \in \mathcal{F}_1$. To conclude, it remains to check that $(\Omega, \mathcal{F}_1, \mu_1)$ is a measure space.

The fact that $\mathcal{F}_1$ is closed under complements is easy: if $E, N \in \mathcal{F}$ with $\mu(N) = 0$ and $M \subset N$ then $(E \cup M)^c = (E \cup N)^c \cup (N \setminus M)$, and $(E \cup N)^c \in \mathcal{F}$ and $N \setminus M \subset N$.

Finally, if $(E_i \cup M_i, i \geq 1)$ are elements of $\mathcal{F}_1$, so each $E_i \in \mathcal{F}_i$ and each $M_i \subset N_i$ for some $N_i \in \mathcal{F}$ with $\mu(N_i) = 0$, then

$$\bigcup_{i \geq 1} E_i \cup M_i = \bigcup_{i \geq 1} E_i \cup \bigcup_{i \geq 1} M_i,$$

and $\bigcup_{i \geq 1} M_i \subset \bigcup_{i \geq 1} N_i$ and $\mu(\bigcup_{i \geq 1} N_i) \leq \sum_{i \geq 1} \mu(N_i) = 0$, so $\bigcup_{i \geq 1} E_i \cup M_i \in \mathcal{F}_1$. □

4. [Half-open intervals] Let

$$\mathcal{A} = \left\{ \bigcup_{i=1}^{n} (a_i, b_i] : n \geq 1, -\infty < a_i \leq b_i < \infty \text{ for all } 1 \leq i \leq n \right\}.$$

Show that $\mathcal{A}$ is a ring.

Proof. To check closure under finite unions it suffices to check that if $A, C \in \mathcal{A}$ then $A \cup C$ in $\mathcal{A}$. So suppose that $A = \bigcup_{i=1}^{n} (a_i, b_i]$ and $C = \bigcup_{j=1}^{m} (c_j, d_j]$ are two elements of $\mathcal{A}$. We assume both sets are expressed as disjoint unions.

For each $i \in [n]$ and $j \in [m]$, if $(a_i, b_i] \cap (c_j, d_j]$ is non-empty we denote the intersection by $(\ell_{ij}, r_{ij}]$. Then

$$A \cup C = \bigcup_{i,j:(a_i,b_i] \cap (c_j,d_j] \neq \emptyset} (\ell_{i,j}, r_{i,j}]\,.$$

thus $A \cup C$ is again a finite union of half-open intervals so $A \cup C \in \mathcal{A}$.

To prove that $A \setminus C$ is in $\mathcal{A}$, we may assume by induction that $j = 1$, or in other words that C contains a single interval $C = \{(c, d]\}$. Moreover, since

$$A \setminus \{(c, d]\} = \bigcup_{i=1}^{n} (a_i, b_i] \setminus (c, d],$$

and we already checked that $\mathcal{A}$ is closed under finite unions, it suffices to verify that each set $(a_i, b_i] \setminus (c, d]$ is in $\mathcal{A}$. But for any two intervals $(a, b]$ and $(c, d]$, either $(a, b] \setminus (c, d]$ is empty, or it consists of one or two disjoint half-open intervals of the same form, so such a difference indeed lies in $\mathcal{A}$. □

5. [An alternate description of finite pre-measures]
Fix a set Ω and a ring $\mathcal{A}$ on it. Let $\mu : \mathcal{A} \to [0,1]$ be an additive function. Show that μ is a pre-measure if and only if for any decreasing sequence $(E_i, i \geq 1)$ of elements of $\mathcal{A}$ with $\bigcap_{i\geq 1} E_i = \emptyset$ we have $\lim_{i\to\infty} \mu(E_i) = 0$.
[That is, a finite additive function on $\mathcal{A}$ is a pre-measure iff it is continuous at $\emptyset$.]

Proof. First suppose μ is a pre-measure on $\mathcal{A}$, and fix a decreasing sequence $(E_i, i \geq 1)$ of elements of $\mathcal{A}$ with $\bigcap_{i\geq 1} E_i = \emptyset$. Let $F_1 = E_1^c$ and for $i \geq 1$ let $F_{i+1} = E_i \setminus E_{i+1}$. Then $(F_i, i \geq 1)$ are disjoint and $\bigcup_{j=1}^{i} F_j = E_i^c$ for all i. Since $\bigcap_{i\geq 1} E_i = \emptyset$, it follows that $\bigcup_{i\geq 1} F_i = \Omega \in \mathcal{A}$, so

$$1 = \mu(\Omega) = \sum_{i\geq 1} \mu(F_i)\,;$$

likewise, $\bigcap_{j=i+1}^{\infty} F_j = E_i$ so

$$\mu(E_i) = \sum_{i=j+1}^{\infty} \mu(F_j).$$

Since $\sum_{i\geq 1} \mu(F_i) = 1$, the sum $\sum_{i=j+1}^{\infty} \mu(F_j)$ must tend to 0 as $j \to \infty$.
Next suppose μ is continuous at 0, and fix any sequence $(A_i, i \geq 1)$ of disjoint elements of $\mathcal{A}$ with $A := \bigcup_{i\geq 1} A_i \in \mathcal{A}$. By finite additivity, $\mu(A) \geq \sum_{j=1}^{i} \mu(A_j)$ for all j, so taking limits $\mu(A) \geq \sum_{i\geq 1} \mu(A_i)$, and we just need to prove the reverse inequality. For this let $E_i = A \setminus \bigcup_{j=1}^{i} A_j$. Then

$$\mu(A) = \sum_{j=1}^{i} \mu(A_j) + \mu(E_i) \leq \sum_{j=1}^{\infty} \mu(A_j) + \mu(E_i).$$

But $E_i \downarrow \emptyset$ as $i \to \infty$, so $\mu(E_i) \to 0$ as $i \to \infty$ by assumption. Optimizing the previous bound over i it follows that $\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j)$. ☐

6. [Carathéodory for measure spaces of finite measure.]
Let $(E, \mathcal{A})$ be a field with $E \in \mathcal{A}$, and let $\mu : \mathcal{A} \to [0,1]$ be a pre-measure on $\mathcal{A}$. For $A, B \subset E$ let $d(A,B) := \mu^*(A \triangle B)$. Here and below,

$$\mu^*(B) := \inf\left(\sum_{n\geq 1} \mu(A_n) : A_n \in \mathcal{A}, n \geq 1; B \subset \bigcup_{n\geq 1} A_n \right)$$

denotes the outer measure generated by μ, and $A \triangle B$ denotes symmetric difference.

(a) Show that d is a finite pseudometric (it satisfies the triangle inequality) on 2^E.
(b) Show that the union and complementation operations are continuous with respect to d.
[That is, if $A, B, C \in \mathcal{A}$, and $(A_n, n \geq 1)$ is a sequence of elements of $\mathcal{A}$ such that $d(A_n, A) \to 0$, then $d(A_n \cup B, C) \to d(A \cup B, C)$ and $d(A_n^c, B) \to d(A^c, B)$.]
(c) Let $\mathcal{B}$ be the closure of $\mathcal{A}$ with respect to the pseudometric d. Prove that $\mathcal{B}$ is a σ-field.
[By closure, we mean that a subset B of E lies in $\mathcal{B}$ if and only if there exists a sequence $(B_n, n \geq 1)$ of elements of $\mathcal{A}$ such that $d(B_n, B) \to 0$.]

(d) Show that for all $A, B \in \mathcal{A}$, $|\mu^*(A) - \mu^*(B)| \leq d(A, B)$. Deduce that μ^* is finitely additive on $\mathcal{B}$.

(e) Show that μ^* is a measure on $\mathcal{B}$.

Proof. (a) This is simply the set-theoretic fact that $A \triangle C \subset (A \triangle B) \cup (B \triangle C)$, together with subadditivity of outer measure.

(b) Fix sets $B, C \in \mathcal{A}$. The let $\epsilon > 0$, and fix any sets $A, A' \in \mathcal{A}$ with $d(A, A') < \epsilon$. Note that

$$(A \cup B) \triangle C \subset ((A' \cup B) \triangle C) \cup (A' \triangle A),$$

so by subadditivity of μ^*,

$$d((A \cup B) \triangle C) \leq d((A' \cup B) \triangle C) + d(A' \triangle A) < d((A' \cup B) \triangle C) + \epsilon.$$

by symmetry it follows that

$$|d((A \cup B) \triangle C) - d((A' \cup B) \triangle C)| < \epsilon,$$

so the union operation is continuous (in fact, Lipschitz continuous with constant 1) with respect to d.

Similarly, since $A^c \triangle B = (A \cap B) \cup (A^c \cap B^c)$, for sets A_1, A_2, B we have

$$\begin{aligned} A_2^c \triangle B &= (A_2 \cap B) \cup (A_2^c \cap B^c) \\ &\subseteq (A_1 \cup B) \cup (A_2 \setminus A_1) \cup (A_1^c \cap B^c) \cup (A_2^c \setminus A_1^c) \\ &= (A_1^c \triangle B) \cup (A_1 \triangle A_2). \end{aligned}$$

Thus if $A_1, A_2, B \in \mathcal{A}$ and $d(A_1, A_2) < \epsilon$ then $d(A_2^c, B) \leq d(A_1^c, B) + d(A_1, A_2)$, and by symmetry it follows that $|d(A_1^c, B) - d(A_2^c, B)| < \epsilon$ so the complementation operation is also continuous with respect to d.

(c) First, $E \in \mathcal{A} \subset \mathcal{B}$. We next check closure under complements. We use that for any sets A and B, $A \triangle B = A^c \triangle B^c$. Fix $B \in \mathcal{B}$ - then there exists a sequence $(A_n, n \geq 1)$ of elements of $\mathcal{A}$ with $d(A_n, B) \to 0$. For such a sequence, we then have $d(A_n^c, B^c) \to 0$, and the sets A_n^c belong to $\mathcal{A}$ since $\mathcal{A}$ is a field; so $B^c \in \mathcal{B}$. It remains to prove that $\mathcal{B}$ is closed under countable unions; as usual, it suffices to prove closure under countable disjoint unions. So fix a sequence $(B_n, n \geq 1)$ of disjoint elements of $\mathcal{B}$. We wish to show that $B := \bigcup_{n \geq 1} \in \mathcal{B}$, or in other words to show that for any $\epsilon > 0$ there exists $A \in \mathcal{A}$ with

$$d(A, B) = \mu^*(A \triangle B) < \epsilon.$$

For this, fix $\epsilon > 0$ and for each $n \geq 1$ choose $A_n \in \mathcal{A}$ with $d(A_n, B_n) < \epsilon/2^n$. Write $A = \bigcup_{n \geq 1} A_n$ and $B = \bigcup_{n \geq 1} B_n$. Then

$$A \triangle B = \left(\bigcup_{n \geq 1} A_n \setminus \bigcup_{n \geq 1} B_n \right) \cup \left(\bigcup_{n \geq 1} B_n \setminus \bigcup_{n \geq 1} A_n \right) \subset \bigcup_{n \geq 1} A_n \triangle B_n,$$

so by subadditivity of μ^*,

$$d(A, B) = \mu^*(A \triangle B) \leq \sum_{n \geq 1} \mu^*(A_n \triangle B_n) = \sum_{n \geq 1} d(A_n, B_n) < \sum_{n \geq 1} \frac{\epsilon}{2^n} = \epsilon.$$

(d) Fix $A, B \in \mathcal{A}$. Since $A \subset B \cup (A \triangle B)$, by subadditivity of outer measure we have $\mu^*(A) \leq \mu^*(B) + \mu^*(A \triangle B)$. By symmetry we likewise have $\mu^*(B) \leq \mu^*(A) + \mu^*(A \triangle B)$, and it follows that

$$|\mu^*(A) - \mu^*(B)| \leq d(A, B).$$

Now fix $B, B' \in \mathcal{B}$ disjoint. Then for all $\epsilon > 0$, we may choose $A, A' \in \mathcal{A}$ with $d(A, B) < \epsilon$ and $d(A', B') < \epsilon$. We saw in class that $\mu = \mu^*$ on $\mathcal{A}$; this is an easy consequence of subadditivity. Since μ is finitely additive on $\mathcal{A}$, it follows that

$$\mu^*(A \cup A') = \mu^*(A) + \mu^*(A') - \mu^*(A \cap A'). \tag{0.1}$$ eq:to_compa

By our choice of A and A', we may replace $\mu^*(A)$ and $\mu^*(A')$ by $\mu^*(B)$ and $\mu^*(B')$ on the right-hand side of (0.1) at the cost of incurring an error in the equation of at most 2ϵ.

Next, since B and B' are disjoint,

$$A \cap A' \subset (A \setminus B) \cup (A' \setminus B') \subset (A \triangle B) \cup (A' \triangle B'),$$

so

$$0 \leq \mu^*(A \cap A') \leq \mu^*((A \triangle B) \cup (A' \triangle B')) \leq \mu^*(A \triangle B) + \mu^*(A' \triangle B') < 2\epsilon.$$

Thus we may remove the term $\mu^*(A \cap A')$ from the right of (**??** and incur an additional error of less than 2ϵ.

Finally, since $(A \cup A') \triangle (B \cup B') \subset (A \triangle B) \cup (A' \triangle B')$, we have

$$d(A \cup A', B \cup B') \leq d(A, B) + d(A', B') < 2\epsilon,$$

so we may replace $\mu^*(A \cup A')$ by $\mu^*(B \cup B')$ on the left of (0.1) and incur a further error of at most 2ϵ. Combining the three error bounds, we obtain that

$$|\mu^*(B \cup B') - \mu^*(B) - \mu^*(B')| \leq 6\epsilon\,;$$

as $\epsilon > 0$ was arbitrary it follows that $\mu^*(B \cup B') = \mu^*(B) + \mu^*(B')$.

(e) It suffices to prove that μ^* is countably additive. So fix disjoint sets $(B_n, n \geq 1)$ in $\mathcal{B}$ and write $B = \bigcup_{n \geq 1} B_n$. By finite additivity, $\mu^*(B) \geq \sum_{n \geq 1}^{m} \mu^*(B_m)$ for all $m \geq 1$, so taking $m \to \infty$ we obtain that $\mu^*(B) \geq \sum_{n \geq 1} \mu^*(B_m)$. By subadditivity of outer measure we also have that

$$\mu^*(B) \leq \sum_{n \geq 1} \mu^*(B_m),$$

so in fact $\mu^*(B) = \sum_{n \geq 1} \mu^*(B_m)$.

□

7. [Fields approximate the σ-fields they generate.]
Let $\mathcal{A}$ be a field over Ω and let μ be a pre-measure on $\mathcal{A}$. Let $(\Omega, \mathcal{F}, \mu)$ be the measure space extending $(\Omega, \mathcal{A}, \mu)$ constructed in proving the Caratheodory extension theorem.

(a) Show that for any $E \in \mathcal{F}$ with $\mu(E) < \infty$, for all $\epsilon > 0$ there exists $A \in \mathcal{A}$ such that $\mu(A \triangle E) < \epsilon$. ____________________

Proof. Fix E as in the question and $\epsilon > 0$. Let $(B_n, n \geq 1)$ be a cover of E with simple sets so that $\mu(B) < \mu(E) + \epsilon$, where $B = \bigcup_{n\geq 1} B_n$. Write $A_n = \bigcup_{i=1}^n B_i$; note that $A_n \in \mathcal{A}$ for all $n \geq 1$. Then $A_n \uparrow B$ so $\mu(A_n) \uparrow \mu(B)$. Now let m be large enough that $A_m \geq \mu(E) - \epsilon$. Then since $E \triangle A_m \subset B \setminus A_m$ we have

$$\mu(E \triangle A_m) \leq \mu(B \setminus A_m) = \mu(B) - \mu(A_m) < 2\epsilon.$$

The result follows. □

(b) Suppose that μ is σ-finite. Show that for any $E \in \mathcal{F}$, for all $\epsilon > 0$ there exists $A \in \mathcal{A}$ such that $\mu(A \triangle E) < \epsilon$.
[Saying μ is σ-finite means there exists a sequence $(E_i, i \geq 1)$ of elements of $\mathcal{F}$ with $\bigcup_{i\geq 1} E_i = \Omega$ and with $\mu(E_i) < \infty$ for all $i \geq 1$.]
This question was incorrect. A counterexample is as follows. Let $\Omega = \mathbb{N}$, let $\mathcal{A} = \{S \subset \mathbb{N} : |S| < \infty \text{ or } |\mathbb{N} \setminus S| < \infty\}$, and let $\mu(S) = |S|$ for $S \in \mathcal{A}$. Then $\mathcal{A}$ is a ring, μ is a pre-measure on $\mathcal{A}$, and $\sigma(\mathcal{A}) = 2^{\mathbb{N}}$, so by the Carathéodory extension theorem there is an extension of μ to $2^{\mathbb{N}}$. Also, $\mathcal{A}$ is a π-system so the extension is unique; so it must be counting measure on $\mathbb{N}$.
Now let $E \subset \mathbb{N}$ be the even numbers. Then no finite or co-finite set is a good approximation to E.

8. [λ-systems which are π-systems are σ-fields]
Let $\mathcal{A}$ be a collection of subsets of a ground set Ω. Show that if $\mathcal{A}$ is both a λ-system and a π-system then it is a σ-field.

Proof. Omitted □

9. [Entanglement.] Let $\Omega = \{1, 2, 3, 4\}$ and let $\mathcal{F}$ be the set of all subsets of Ω. Find a set $\mathcal{S}$ of subsets of Ω with $\sigma(\mathcal{S}) = \mathcal{F}$, and two distinct probability measures μ, ν on $(\Omega, \mathcal{F})$, such that $\mu(S) = \nu(S)$ for all $S \in \mathcal{S}$.

Proof. Fix $x, y \geq 0$ with $x + y = 1/2$. Let $\mu(1) = \mu(3) = x$ and $\mu(2) = \mu(4) = y$. Then take $\mathcal{S} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$. Note that $\mu(S) = 1/2$ for all $S \in \mathcal{S}$, whatever the values of x and y. □