Math 587 – Fall 2019 – Assignment 1

Assigned on September 5, 2019.
Due on September 16, 2019 at 2 PM (by email or in my office)

This assignment about basic properties of rings, fields, measurable spaces, measure spaces and measures.

1. [Restrictions of measures.] Let $(\Omega, \mathcal{F})$ be a measurable space. Fix $E \subset \Omega$ and let

$$\mathcal{F}|_E := \{F \cap E : F \in \mathcal{F}\}.$$

   Show that $\mathcal{F}|_E$ is a σ-field.

2. [Basic properties of measures]
   Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

   (a) Show that if $E, F \in \mathcal{F}$ and $E \subset F$ then $\mu(E) \le \mu(F)$. (Note: prove this assuming only that μ is countably additive and that $\mu(\emptyset) = 0$.)

   (b) Show that if $(E_i, i \ge 1)$ is an increasing sequence of elements of $\mathcal{F}$, in that $E_i \subseteq E_j$ for $i \le j$, then $\mu(\bigcup_{i\ge 1} E_i) = \lim_{i\to\infty} \mu(E_i)$.

   (b) Show that if $(E_i, i \ge 1)$ is a decreasing sequence of elements of $\mathcal{F}$ and $\mu(E_1) < \infty$ then then $\mu(\bigcap_{i\ge 1} E_i) = \lim_{i\to\infty} \mu(E_i)$.

3. [Null sets]
   Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Say $N \in \mathcal{F}$ is a *null set* if $\mu(N) = 0$. Say that $(\Omega, \mathcal{F}, \mu)$ is *complete* if for any null set N, for all $M \subset N$ we have $M \in \mathcal{F}$. Write $\overline{\mathcal{F}} := \bigcap_{\mathcal{F}' \supset \mathcal{F} : \mathcal{F}' \text{ complete}} \mathcal{F}'$. Prove that

$$\overline{\mathcal{F}} = \{E \cup M : E \in \mathcal{F}, \text{ there exists a null set } N \in \mathcal{F} \text{ such that } M \subset N\}.$$

4. [Half-open intervals] Let

$$\mathcal{A} = \left\{ \bigcup_{i=1}^{n} (a_i, b_i] : n \ge 1, -\infty < a_i \le b_i < \infty \text{ for all } 1 \le i \le n \right\}.$$

   Show that $\mathcal{A}$ is a ring.

5. [An alternate description of finite pre-measures]
   Fix a set Ω and a ring $\mathcal{A}$ on it. Let $\mu : \mathcal{A} \to [0, 1]$ be an additive function. Show that μ is a pre-measure if and only if for any decreasing sequence $(E_i, i \ge 1)$ of elements of $\mathcal{A}$ with $\bigcap_{i\ge 1} E_i = \emptyset$ we have $\lim_{i\to\infty} \mu(E_i) = 0$.
   [That is, a finite additive function on $\mathcal{A}$ is a pre-measure iff it is continuous at $\emptyset$.]

6. [Carathéodory for measure spaces of finite measure.]
   Let $(E, \mathcal{A})$ be a field with $E \in \mathcal{A}$, and let $\mu : \mathcal{A} \to [0,1]$ be a pre-measure on $\mathcal{A}$. For $A, B \subset E$ let $d(A,B) := \mu^*(A \triangle B)$. Here and below,

$$\mu^*(B) := \inf\left(\sum_{n\geq 1} \mu(A_n) : A_n \in \mathcal{A}, n \geq 1; B \subset \bigcup_{n\geq 1} A_n\right)$$

   denotes the outer measure generated by μ, and $A \triangle B$ denotes symmetric difference.

   (a) Show that d is a finite pseudometric (it satisfies the triangle inequality) on 2^E.
   (b) Show that the union and complementation operations are continuous with respect to d.
       [That is, if $A, B, C \in \mathcal{A}$, and $(A_n, n \geq 1)$ is a sequence of elements of $\mathcal{A}$ such that $d(A_n, A) \to 0$, then $d(A_n \cup B, C) \to d(A \cup B, C)$ and $d(A_n^c, B) \to d(A^c, B)$.]
   (c) Let $\mathcal{B}$ be the closure of $\mathcal{A}$ with respect to the pseudometric d. Prove that $\mathcal{B}$ is a σ-field.
       [By closure, we mean that a subset B of E lies in $\mathcal{B}$ if and only if there exists a sequence $(B_n, n \geq 1)$ of elements of $\mathcal{A}$ such that $d(B_n, B) \to 0$.]
   (d) Show that μ^* for all $A, B \in \mathcal{A}$, $|\mu^*(A) - \mu^*(B)| \leq d(A,B)$. Deduce that μ^* is finitely additive on $\mathcal{B}$.
   (e) Show that μ^* is a measure on $\mathcal{B}$.

7. [Fields approximate the σ-fields they generate.]
   Let $\mathcal{A}$ be a field over Ω and let μ be a pre-measure on $\mathcal{A}$. Let $(\Omega, \mathcal{F}, \mu)$ be the measure space extending $(\Omega, \mathcal{A}, \mu)$ constructed in proving the Caratheodory extension theorem.

   (a) Show that for any $E \in \mathcal{F}$ with $\mu(E) < \infty$, for all $\epsilon > 0$ there exists $A \in \mathcal{A}$ such that $\mu(A \triangle E) < \epsilon$.
   (b) Suppose that μ is σ-finite. Show that for any $E \in \mathcal{F}$, for all $\epsilon > 0$ there exists $A \in \mathcal{A}$ such that $\mu(A \triangle E) < \epsilon$.
       [Saying μ is σ-finite means there exists a sequence $(E_i, i \geq 1)$ of elements of $\mathcal{F}$ with $\bigcup_{i\geq 1} E_i = \Omega$ and with $\mu(E_i) < \infty$ for all $i \geq 1$.]

8. [λ-systems which are π-systems are σ-fields]
   Let $\mathcal{A}$ be a collection of subsets of a ground set Ω. Show that if $\mathcal{A}$ is both a λ-system and a π-system then it is a σ-field.

9. [Entanglement.] Let $\Omega = \{1,2,3,4\}$ and let $\mathcal{F}$ be the set of all subsets of Ω. Find a set $\mathcal{S}$ of subsets of Ω with $\sigma(\mathcal{S}) = \mathcal{F}$, and two distinct probability measures μ, ν on $(\Omega, \mathcal{F})$, such that $\mu(S) = \nu(S)$ for all $S \in \mathcal{S}$.