Math 587 – Notes for the first class

L. Addario-Berry

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1 Random Walks

Let $X_1, X_2, \ldots$ be independent and identically distributed integer random variables, let $S_0 = 0$ and let $S_n = \sum_{i=1}^{n} X_i$ for each $n > 0$. We say that $(S_n)_{n\in\mathbb{N}}$ is a *random walk with step size* X. The idea is that at each unit of time, we look around and make a random choice of where to go next.

The simplest and most famous example is when the X_i are ± 1-valued random variables. In this case the "location" of the walk on the number line only changes by one at any step, and the walk is called "simple random walk". If $\mathbb{P}\{X_i = 1\} = \mathbb{P}\{X_i = -1\} = 1/2$ then the walk is called "the drunkard's walk" or "symmetric simple random walk".

Simple random walk was first investigated by Pólya in the early 1920's. He posed the basic question: is the drunkard (who stumbles from place to place without heed for where he is going) guaranteed to eventually find his way back home? In less poetic terms, will the random walk eventually return to zero with probability one?

There is a beautiful way to prove that the answer is "yes" for the drunkard's walk using a connection between random walks and electrical networks. (To read about this connection, take a look at the freely downloadable book "Random walks and electric networks" by Peter G. Doyle and J. Laurie Snell.) However, we will prove the result using *generating functions*, one of the fundamental tools in combinatorial probability, and one that also allows us to prove that the answer is "no" if the walk is biased to the right or the left.

Let $p = \mathbb{P}\{X_1 = 1\} = \mathbb{P}\{X_i = 1\}$ for all i. Write $z_n = \mathbb{P}\{S_n = 0\}$, and write $f_n = \mathbb{P}\{S_1 \neq 0, S_2 \neq 0, \ldots, S_{n-1} \neq 0, S_n = 0\}$. (The "z" stands for zero and the "f" stands for first return.) We write

$$P(s) = \sum_{n=0}^{\infty} z_n s^n \qquad \text{and} \qquad F(s) = \sum_{n=0}^{\infty} f_n s^n$$

These are the *generating functions* for the sequences $\{z_n\}_{n\geq 0}$ and $\{f_n\}_{n\geq 1}$. We think of these as formal power series but note that they each have radius of convergence at least 1 since the z_n and the f_n are all at most one.

We may always think of the random walk as consisting of visits to zero separated by excursions away from zero. This point of view is captured in terms of generating functions by the following proposition.

Proposition 1. *P(s) = 1+P(s)F(S)*

Proof. Write A_n for the event that $S_n = 0$, and write B_k for the event that the first time the random walk returns is at the k'th step. The B_k are disjoint and so for $n \geq 1$ we have

$$\mathbb{P}\{A_n\} = \sum_{k=1}^{n} \mathbb{P}\{A_n \mid B_k\}\mathbb{P}\{B_k\}$$

We have $\mathbb{P}\{B_k\} = f_k$, and also $\mathbb{P}\{A_n \mid B_k\} = z_{n-k}$. So we may rewrite the preceding equation as

$$z_n = \sum_{k=1}^{n} z_{n-k} f_k.$$

Now multiply both sides by s^n:

$$z_n s^n = \sum_{k=1}^{n} (z_{n-k} s^{n-k})(f_k s^k).$$

If we sum from $n = 0$ to infinity on the left we obtain $P(s)$. Since $z_0 s^0 = 1$, from the above equality we then obtain

$$P(s) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} (z_{n-k} s^{n-k})(f_k s^k) = 1 + P(s)F(s).$$

□

Corollary 1.

$$P(s) = \frac{1}{\sqrt{1 - 4p(1-p)s^2}} \qquad \textit{and} \qquad F(s) = 1 - \sqrt{1 - 4p(1-p)s^2}.$$

Proof. For the first equation, we calculate z_n by hand. First, n must be even if $S_n = 0$ is to occur. Next, for $S_n = 0$ to occur the random walk must make $n/2$ positive steps and $n/2$ negative steps. For any given such combination of steps, the probability is $(p(1-p))^{n/2}$, and there are $\binom{n}{n/2}$ such combinations, so for n even we have

$$z_n = \binom{n}{n/2}(p(1-p))^{n/2}.$$

So, establishing the first equation reduces to showing that

$$\sum_{m=0}^{\infty} \binom{2m}{m}(p(1-p))^m s^{2m} = \frac{1}{\sqrt{1-4p(1-p)s^2}},$$

which is just a matter of writing out the Taylor series for $(1-x)^{-1/2}$.

Next, from Proposition 1, we have

$$F(s) = \frac{P(s)-1}{P(s)} = 1 - \frac{1}{P(s)} = 1 - \sqrt{1-4p(1-p)s^2}.$$

□

From the corollary, we can immediately derive the answer to Polya's question.

Theorem 2. *The probability that the random walk ever returns to the origin is* $1-|2p-1|$.

Proof. By the definition of $F(s)$, the probability that the random walk returns to the origin is

$$F(1) = \sum_{n=1}^{\infty} \mathbb{P}\{\tau = n\} = 1 - \sqrt{1-4p(1-p)} = 1 - \sqrt{(2p-1)^2} = 1 - |2p-1|.$$

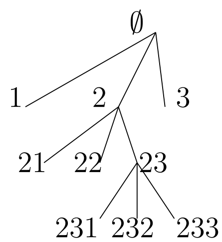
□

Corollary 2. *If* $p = 1/2$ *then for any integer* $k \leq 0$, *the probability that the random walk ever visits* k *is* 1.

Proof. Each time the random walk returns to the origin, it has another chance to walk from the origin to k. It will visit the origin an infinite number of times, so eventually it must visit k (in fact, also an infinite number of times). □

2 Branching Processes (Galton-Watson Trees)

Fix a random variable $B, B \in \mathbb{N}$. (B is the *branch factor*)



⋆ Start from the root (call it $\emptyset$), let $B_\emptyset \sim B$.
⋆ Give $\emptyset$ children $1, \ldots, B_\emptyset$.
⋆ Independently for each $i = 1, \ldots, B_\emptyset$, let $B_i \sim B$, and
⋆ give i children $i1, i2, \ldots, iB_i$.
⋆ Repeat forever or until done; call the resulting random tree $\mathcal{T}_B$.

Let Z_n be the number of individuals in the n'th generation (the individuals of the n'th generation are those whose name is n characters long), and write $|\mathcal{T}_B| = \sum_{n=0}^{\infty} Z_n$ for the total number of individuals. We say the survival occurs if $Z_n > 0$ for all n, and otherwise that say that extinction occurs. Equivalently, survival occurs if $|\mathcal{T}_B| = \infty$, and extinction occurs if $|\mathcal{T}_B| < \infty$.

Theorem 3 (Fundamental theorem of branching processes)**.** $\mathbb{P}\{|\mathcal{T}_B| = \infty\} > 0$ *if and only if one of the following two conditions holds.*

- $\mathbb{P}\{B = 1\} = 1$
- $\mathbb{E}\{B\} > 1$.

As a warm up, we prove the following lemma.

Lemma 1. *For all n, $\mathbb{E}\{Z_n\} = [\mathbb{E}\{B\}]^n$.*

Proof. This is obviously true for $n = 0$. Supposing the equality holds for a given n, we write

$$\mathbb{E}\{Z_{n+1}\} = \sum_{i=0}^{\infty} \mathbb{E}\{Z_{n+1} \mid B_\emptyset = i\} \mathbb{P}\{B_\emptyset = i\}.$$

Given that $B_\emptyset = i$, the children $1, \ldots, i$ of $\emptyset$ are each the root of an independent copy of the whole process, so

$$\mathbb{E}\{Z_{n+1} \mid B_\emptyset = i\} = i\mathbb{E}\{Z_n\}.$$

We thus have

$$\mathbb{E}\{Z_{n+1}\} = \sum_{i=0}^{\infty} i\mathbb{E}\{Z_n\}\mathbb{P}\{B_\emptyset = i\} = \mathbb{E}\{Z_n\} \cdot \mathbb{E}\{B\} = [\mathbb{E}\{B\}]^{n+1},$$

the last step by induction. □

Corollary 3. *If* $\mathbb{E}\{B\} < 1$ *then* $\mathbb{E}\{|\mathcal{T}_B|\} < \infty$*, so* $\mathbb{P}\{|\mathcal{T}_B| = \infty\} = 0$*.*

Proof. If $\mathbb{E}\{B\} < 1$ then

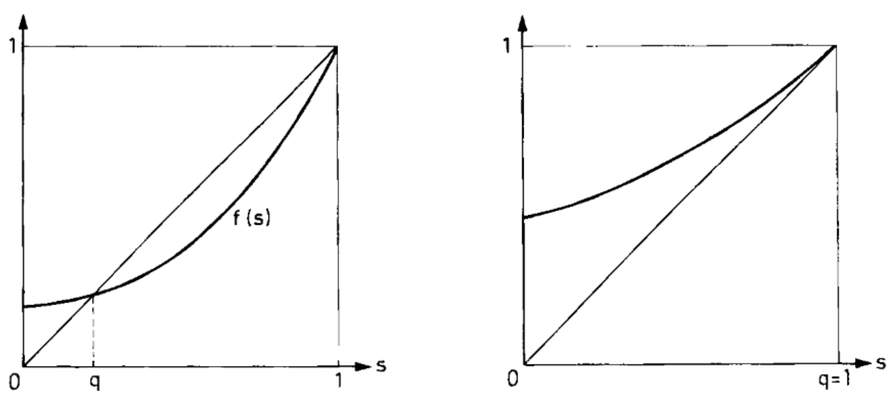
$$\mathbb{E}\{|\mathcal{T}_B|\} = \sum_{n=0}^{\infty} \mathbb{E}\{Z_n\} = \sum_{n=0}^{\infty} \mathbb{E}\{B\}^n = \frac{1}{1 - \mathbb{E}\{B\}} < \infty.$$

It follows by Markov's inequality that $\mathbb{P}\{|\mathcal{T}_B| = \infty\} = 0$. □

Now let $F(z) = \mathbb{E}\{z^B\} = \sum_{k=0}^{\infty} \mathbb{P}\{B = k\} z^k$.

Proposition 4 (Fundamental theorem of branching processes)**.** *If* $\mathbb{P}\{B = 1\} < 1$ *then*

$$\mathbb{P}\{|\mathcal{T}_B| < \infty\} = \min_{x \geq 0}\{F(x) = x\}.$$



Proof. Write $p = \mathbb{P}\{|\mathcal{T}_B| < \infty\}$. We prove the proposition in two parts: first we show that $F(p) = p$, and second we show that p is the *smallest* non-negative solution of $F(x) = x$.

The proof of the first part is similar to that of the proof of the lemma. We begin by noting that

$$|\mathcal{T}_B| < \infty \Leftrightarrow Z_n = 0 \text{ for some } n,$$

so

$$p = \mathbb{P}\{\bigcup_{n=0}^{\infty} Z_n = 0\}.$$

The events on the right are increasing (if $Z_n = 0$ then $Z_{n+1} = 0$) so it follows that

$$p = \lim_{n\to\infty} \mathbb{P}\{Z_n = 0\}.$$

Now write $F_1(x) = F(x)$ and for $n > 1$ write $F_n(x) = F(F_{n-1}(x))$, so $F_n(x)$ is the result of applying F to x n times.

We claim that for all $n \geq 1$, $\mathbb{P}\{Z_n = 0\} = F_n(0)$. When $n = 1$, we have $F_1(0) = F(0) = \mathbb{P}\{B = 0\} = \mathbb{P}\{Z_1 = 0\}$. For larger n, we apply the same inductive technique as in Lemma 1.

$$\begin{aligned}
\mathbb{P}\{Z_n = 0\} &= \sum_{i=0}^{\infty} \mathbb{P}\{Z_n = 0 \mid Z_1 = i\}\mathbb{P}\{Z_1 = i\} \\
&= \sum_{i=0}^{\infty} \mathbb{P}\{Z_{n-1} = 0\}^i \mathbb{P}\{B = i\} \\
&= \sum_{i=0}^{\infty} F_{n-1}(0)^i \mathbb{P}\{B = i\} \\
&= F(F_{n-1}(0)) \\
&= F_n(0).
\end{aligned}$$

We now have

$$p = \lim_{n\to\infty} F_n(0).$$

Since $F_n(0) \to p$ and F is continuous, we also have $F(F_n(0)) \to F(p)$. But $F(F_n(0)) \to p$, so we must have $p = F(p)$.

For the second part, suppose q is any other non-negative solution of $F(x) = x$. By differentiation we see that F is non-decreasing and so since $q \geq 0$ we must have $q = F(q) \geq F(0)$. Repeatedly applying F we see that we must have $q \geq F_n(0)$ for all n, and so $q \geq \lim_{n\to\infty} F_n(0) = p$. $\square$

Proof of Fundamental Theorem. We already saw that if $\mathbb{E}\{B\} < 1$ then extinction is certain, so we assume that $\mathbb{E}\{B\} \geq 1$. Case (a) is also obvious so we assume that $\mathbb{P}\{B = 1\} < 1$. Note that $F(0) = \mathbb{P}\{B = 0\} \geq 0$ and that $F''(x) > 0$ for all $x > 0$. Also,

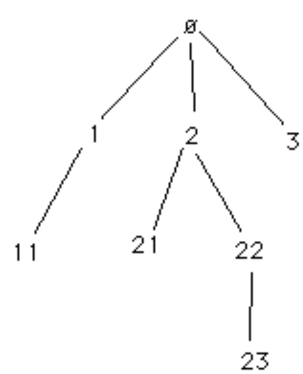
$$F'_B(z) = (\sum_{n=0}^{\infty} \mathbb{P}\{B = n\} z^n)' = \sum_{n=1}^{\infty} n\mathbb{P}\{B = n\} z^{n-1},$$

so $F'_B(1) = \sum_{n=1}^{\infty} n\mathbb{P}\{B = n\} = \mathbb{E}\{B\}$. If $\mathbb{E}\{B\} > 1$ then by continuity there is $x < 1$ such that $F(x) < x$, so by the intermediate value theorem, there is $0 \leq y < x$ with $F(y) = y$, and we must have $p < 1$. On the other hand, if $\mathbb{E}\{B\} = 1$ then by $F''(x) > 0$ for all $x > 0$ we must have $F(x) > x$ for all $0 \leq x < 1$ and so $p = 1$. $\square$

3 Studying branching processes using random walks

Exploration procedure for trees. (Rooted, ordered trees – nodes labeled by finite strings of integers as above.)
$\mathcal{Q} \longrightarrow$ set of seen but unexplored vertices.
"Explore a vertex" $\longrightarrow$ reveal its children.

Example



$\mathcal{Q} = \{\emptyset\}$
$\mathcal{Q} = \{1, 2, 3\}$
$\mathcal{Q} = \{11, 2, 3\}$
$\mathcal{Q} = \{21, 22, 2, 3\}$
$\mathcal{Q} = \{22, 2, 3\}$
$\mathcal{Q} = \{23, 2, 3\}$
$\mathcal{Q} = \{2, 3\}$
$\mathcal{Q} = \{3\}$
$\mathcal{Q} = \emptyset$

DEPTH FIRST SEARCH Start with $\mathcal{Q} = \{\emptyset\}$. **While** $\mathcal{Q} \neq \emptyset$ **1.** Let v be the first element of $\mathcal{Q}$. **2.** Remove v from $\mathcal{Q}$, and add the children of v to the front of $\mathcal{Q}$ in right-to-left order (left one ends up on the front).

When $\mathcal{T} = \mathcal{T}_B$ is a Galton-Watson tree, the change in queue length at one step is distributed as $B-1$, independently of all previous steps. The following lemma is immediate. Let $S_0 = 0$, $S_i = 1 + \sum_{k=1}^{i} X_k \ \forall i$, and $X_1, X_2, \ldots$ iid and distributed as $B - 1$.

Lemma 2. *For all n, $\mathbb{P}\{|\mathcal{T}_B| = n\} = \mathbb{P}\{S_n = -1, S_i \geq 0 \ \forall \ 0 \leq i < n\}$.*

Corollary 4. *If if $\mathbb{P}\{B = 2\} = p$ and $\mathbb{P}\{B = 0\} = 1 - p$, then $\mathbb{P}\{|\mathcal{T}_B| = \infty\} > 0$ if $p > 1/2$ and $\mathbb{P}\{|\mathcal{T}_B| = \infty\} = 0$ if $p = 1/2$.*

Proof. We have

$$\begin{aligned}
\mathbb{P}\{|\mathcal{T}_B| < \infty\} &= \sum_{n=1}^{\infty} \mathbb{P}\{|\mathcal{T}_B| = n\} \\
&= \sum_{n=1}^{\infty} \mathbb{P}\{S_n = -1, S_i \geq 0 \ \forall \ 0 \leq i < n\} \\
&= \mathbb{P}\{\text{The random walk eventually visits } 0\}.
\end{aligned}$$

We saw in the first section (Corollary 2) that this probability is one if $p = 1/2$. On the other hand, if $p > 1/2$ then the probability that the random walk ever returns to its starting point is less than 1, and the random walk tends in the positive direction (since in this case the average step size $\mathbb{E}\{X\} = 2p - 1 > 0$). From this it is not hard to show that the probability that the random walk eventually visits -1 is less than one. □

4 Questions

1. What *is* a "sequence of iid random variables"?

2. We can "rigorously" define Z_n by using a doubly infinite array of random variables: $(B_k^n)_{n,k\geq 1}$ and setting
$$Z_{n+1} = \sum_{k=1}^{Z_n} B_k^{n+1}\,,$$
but we first need to rigorously define a doubly-infinite array of random variables.

3. The use of conditional expectation above seems unproblematic; but what if individuals had mass, and we were interested in the *mass* M_n of a given generation. If the mass can take arbitrary values in $[0, \infty)$ then we couldn't write
$$\mathbb{E}\{M_{n+1}\} = \sum_{x\geq 0} \mathbb{E}\{M_{n+1}|M_n = x\} \cdot \mathbb{P}\{M_n = x\};$$
in general we may have $\mathbb{P}\{M_n = x\} = 0$, the sum may be uncountable, and conditioning on $M_n = x$ may make no sense. The "solution" is to write
$$\mathbb{E}\{M_{n+1}\} = \mathbb{E}\{\mathbb{E}\{M_{n+1}|M_n\}\},$$
where the inner expectation "treats M_n as though it is fixed rather than random"; and the second expectation "remembers" that M_n is random and averages over M_n. but of course, none of this is rigorous...

4. In the proof of the fundamental theorem of branching processes, we used that since
$$\{Z_1 = 0\} \subset \{Z_2 = 0\} \subset \ldots \subset \{Z_n = 0\} \subset \{Z_{n+1} = 0\} \subset \ldots\,,$$
and so
$$\mathbb{P}\{\lim_{n=0}^{\infty}]\{Z_n = 0\}\} = \lim_{n\to\infty} \mathbb{P}\{Z_n = 0\}\,.$$
This implication must be proved!

5. An argument similar to that of Lemma 1 shows that $\mathbb{E}\{Z_{n+1}|Z_0, \ldots, Z_n\} = \mathbb{E}\{B\} \cdot \mathbb{E}\{Z_n\}$, where here again our conditional expectation "treats $Z_0, \ldots, Z_n$ as fixed". This equality states that $\{Z_n/[\mathbb{E}\{B\}]^n\}_{n\geq 0}$ *is a martingale*; the non-negative Martingale convergence theorem then says that
$$\frac{Z_n}{[\mathbb{E}\{B\}]^n} \to W$$

where the convergence is *almost sure* (occurs with probability one) and the limit W is some random variable. Since the left hand side has expectation 1 for all n (by Lemma 1), we may expect that $\mathbb{E}\{W\} = 1$; however, if $\mathbb{E}\{B\} \leq 1$ we already know that $Z_n \to 0$ and so in fact $\mathbb{P}\{W = 0\} = 1$. What is going on?

6. It turns out that $\mathbb{E}\{W\} = 1$ if and only if $\mathbb{E}\{W\} > 1$ and $\mathbb{E}\{L \log(L+1)\} < \infty$; this is not at all "intuitively obvious". In fact, the only immediate implication is given by Fatou's lemma, which states that for any sequence of random variables $\{X_n\}_{n \geq 1}$,

$$\mathbb{E}\{\liminf_{n\to\infty} X_n\} \leq \liminf_{n\to\infty} \mathbb{E}\{X_n\}\,,$$

and so we must have $\mathbb{E}\{W\} \leq 1$ in all cases. (However: what is the lim inf of a sequence of random variables, anyway?)

Homework: read the concrete example from chapter 0 of the book; it is a little different from ours and highlights different questions and intuitions.