Winter 2018 – Math 547 – MIDTERM 2 – March 22, 8 :35-9 :55

Read questions **carefully** before answering. Justify your answers. State any result you use from class.

There are 17 points possible, but the exam will be marked out of 15.

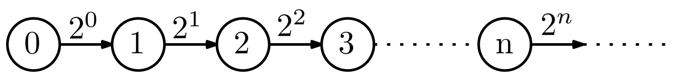

FIGURE 1 – A continuous time Markov chain

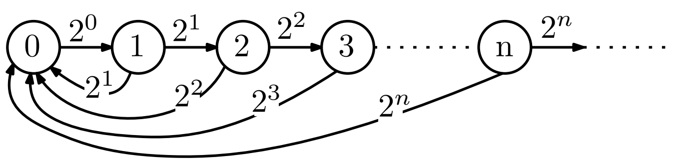

FIGURE 2 – Another continuous-time Markov chain

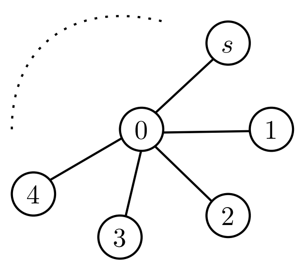

FIGURE 3 – The star with s leaves

1. Recall the notation T_n for the n'th jump time of a continuous-time Markov chain, and $T_\infty = \lim_{n\uparrow\infty} T_n$. A continuous-time Markov chain *explodes* if $\mathbb{P}(T_\infty < \infty) > 0$. The chains in Figures 1 and 2 have all non-zero jump rates indicated.

1 point a) Does the chain in Figure 1 explode ?

Answer. Yes. We saw in class that for a pure birth chain with birth rates $(\lambda_i, i \geq 0)$ a necessary and sufficient condition for the chain to explode is that $\sum_{i\geq 0}(1/\lambda_i) < \infty$. In this case $\lambda_i = 2^i$ so the sum is clearly finite.

1 point b) Does the chain in Figure 2 explode ?

Answer. No. Note that the jump chain of this Markov chain has transition probabilities $p_{0,1} = 1$ and $p_{i,0} = p_{i,i+1} = 1/2$ for $i \geq 1$. In other words, the jump chain is the slippery ladder, which is recurrent. We saw in class (and in Theorem 6.8.24 of the book) that if a continuous time random walk starts at a recurrent state of the jump chain, then it does not explode.

1 points c) What is $\lim_{t\uparrow T_\infty} \mathbb{P}(X_t = 3 \mid X_0 = 0)$ for the chain in Figure 1?

Answer. This question was poorly phrased in two ways. First, it is not clear what $\mathbb{P}(X_t = 3 \mid X_0 = 0)$ means, because it is not clear how to interpret X_t if $t \geq T_\infty$. To fix this, a reasonable assumption would be that $\mathbb{P}(X_t = 3 \mid X_0 = 0)$ means $\mathbb{P}(T_\infty > t, X_t = 3 \mid X_0 = 0)$.

Let E_i be Exponential(2^i) for $i \geq 0$, with the E_i mutually independent. Then

$$\mathbb{P}(T_\infty > t, X_t = 3 \mid X_0 = 0) = \mathbb{P}(E_0 + E_1 + E_2 \leq t < E_0 + E_1 + E_2 + E_3).$$

$$\int_0^t \int_0^{t-x} \int_0^{t-x-y} e^{-x}(2e^{-2y})(4e^{-4z} e^{-8(t-x-y-z)} dx = \frac{8}{21}e^{-t} - \frac{2}{3}e^{-2t} + \frac{1}{3}e^{-4t} - \frac{1}{21}e^{-8t},$$

if my calculus is right.

The second confusing point is that we are taking $\lim_{t\uparrow T_\infty}$, and T_∞ is a random variable. If we just take the limit in the above expression we get

$$\frac{8}{21}e^{-T_\infty} - \frac{2}{3}e^{-2T_\infty} + \frac{1}{3}e^{-4T_\infty} - \frac{1}{21}e^{-8T_\infty}.$$

so this is a "random probability". This expression is one reasonable possible answer to the question.

Another would be to take the expected value of the above random variable, though I don't see a trivial way to compute this.

But in fact, what I actually meant to get at was that because the chain explodes, it must make a "last visit" to state 3, and so

$$\lim_{\epsilon \to 0} \mathbb{P}(X_{T_\infty - \epsilon} = 3 \mid X_0 = 0) = 0.$$

Proving that this last limit is zero is straightforward, since it can be bounded by $\mathbb{P}(T_4 - T_3 < \epsilon)$ which tends to 0 as $\epsilon \to 0$.

1 point d) What is $\lim_{t\uparrow T_\infty} \mathbb{P}(X_t = 3 \mid X_0 = 0)$ for the chain in Figure 2?

Answer. Since this chain is positive recurrent, the limit is just given by the stationary distribution $\pi(3)$. The forward equation gives that $\pi(n+1) = \pi(n)/4$ for all $n \geq 0$. It follows that $\pi(n) = 4^{1-n}/3$ for all n, so $\pi(3) = 1/48$.

2 points e) Let $\tau(n) = \inf\{t : X_t = n\}$. Find simple formulas for $\mathbb{E}[\tau(n) \mid X_0 = 0]$ for the chains in Figure 1 and Figure 2.

Answer. For the chain in figure 1, just use that $\mathbb{E}(T_{n+1} - T_n) = 2^{-n}$ and the fact for a pure birth chain, such as this one, $\tau(n) = T_n$. This implies that $\mathbb{E}(\tau(n)) = 2(1 - 2^{-n})$.

For the chain in figure 2, work out a recurrence describing $\mathbb{E}[\tau(n+1) \mid X_0 = n]$. The chain leaves state n at rate $2^n + 2^n = 2^{n+1}$, so the expected time until the chain moves given that $X_0 = n$ is $1/2^{n+1}$, When it moves, with probability $1/2$ it goes to

$n+1$ and with probability 1/2 it goes to 0. Using the Markov property, this implies that

$$\begin{aligned}\mathbb{E}[\tau(n+1) \mid X_0 = n] &= \mathbb{E}[T_1 \mid X_0 = n] \cdot \mathbb{P}(X_{T_1} = n+1 \mid X_0 = n) \\ &\quad + (\mathbb{E}[T_1 \mid X_0 = n] + \mathbb{E}[\tau(n+1) \mid X_0 = 0]) \cdot \mathbb{P}(X_{T_1} = 0 \mid X_0 = n) \\ &= \frac{1}{2^{n+1}} + \frac{1}{2}\mathbb{E}[\tau(n+1) \mid X_0 = 0]) \, .\end{aligned}$$

Since to get to $n+1$ from 0 the chain must first visit n, then get to $n+1$ from n, it follows that

$$\mathbb{E}[\tau(n+1) \mid X_0 = 0] = \mathbb{E}[\tau(n) \mid X_0 = 0] + \frac{1}{2^{n+1}} + \frac{1}{2}\mathbb{E}[\tau(n+1) \mid X_0 = 0])$$

so

$$\mathbb{E}[\tau(n+1) \mid X_0 = 0] = 2\mathbb{E}[\tau(n) \mid X_0 = 0] + \frac{1}{2^n} \, .$$

Solving this recurrence yields $\mathbb{E}[\tau(n) \mid X_0 = 0] = 2^{n+1}(1 - 4^{-n})/3$.

2. The lazy random walk on the star with s leaves (the graph in Figure 3) is the Markov chain on $\{0, 1, \ldots, s\}$ with $p_{ii} = 1/2$ for $0 \le i \le s$, with $p_{0i} = 1/(2s)$ for $1 \le i \le s$, and with $p_{i0} = 1/2$ for $1 \le i \le s$.

   This question asks you to study a particular coupling between two lazy SRWs on the star with s leaves. More precisely, consider the Markov chain $((Y_n, Z_n), n \ge 0)$ which moves as follows.

   If $Y_n \neq Z_n$ then exactly one of Y_n and Z_n moves, each with equal probability. Whichever coordinate moves behaves like *non-lazy* simple random walk on the star with s leaves.

   If $Y_n = Z_n$ then $Y_{n+1} = Z_{n+1}$ and the coordinates together behave like *lazy* simple random walk on the star with s leaves.

   To make everything fully explicit, the transition probabilities are fully specified as follows.

$$\mathbb{P}_{(0,0)}((Y_1, Z_1) = (i,i)) = \begin{cases} 1/2 & i = 0 \\ 1/(2s) & i \neq 0 \end{cases}$$

$$\mathbb{P}_{(i,i)}((Y_1, Z_1) = (j,j)) = \begin{cases} 1/2 & i = j \\ 1/2 & j = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}_{(i,j)}((Y_1, Z_1) = (i,\ell)) = \mathbb{P}_{(j,i)}((Y_1, Z_1) = (\ell,i)) = \begin{cases} 1/2 & j \neq 0, \ell = 0 \\ 1/(2s) & j = 0, \ell \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

1 point (a) Is the chain $((Y_n, Z_n), n \ge 0)$ irreducible ?

Answer. It is not irreducible ; once the chain visits a state on the diagonal $\Delta = \{(i,i), 0 \le i \le s\}$ it can never leave the diagonal.

1 point (b) What is its stationary distribution ?

Answer. Once the chain reaches the diagonal it just behaves like lazy simple random walk on the star. The stationary distribution for simple random walk on the star gives mass $1/2$ to the centre and $1/(2s)$ to each leaf. Making a random walk lazy doesn't change its stationary distribution (exercise !), so the stationary distribution of the coupled chain is $\pi((0,0)) = 1/2$ and $\pi((i,i)) = 1/(2s)$ for $1 \le i \le s$.

1 point (c) Is it reversible ?

Answer. It is reversible ; just verify that $\pi(x)p_{xy} = \pi(y)p_{yx}$ for all $x, y \in \{0, 1, \ldots, s\}^2$. This holds trivially if x or y is not on the diagonal since in this case both sides equal zero. For the diagonal states, the verification is easy.

2 points (d) Show that $(Y_n, n \ge 0)$ is lazy simple random walk on the star with s leaves.

Answer. Write p_{uv} for the probability that lazy SRW on the star with s leaves moves from u to v in one step. In other words, p_{uv} is $1/2$ if $v = u$, $1/2$ if $v = 0$ and $u \ne 0$, and is $1/(2s)$ if $v \ne 0$ and $u = 0$.

You need to show that

$$\mathbb{P}(Y_{n+1} = y \mid Y_i = y_i, 0 \le i \le n) = \mathbb{P}(Y_{n+1} = y \mid Y_n = y_n) = p_{y_n y}.$$

For this, note that (writing $i \le n$ instead of $0 \le i \le n$ for compactness)

$$\begin{aligned}&\mathbb{P}(Y_{n+1} = y \mid Y_i = y_i, i \le n)\\ &= \sum_{z_0,\ldots,z_n} \mathbb{P}(Y_{n+1} = y \mid (Y_i, Z_i) = (y_i, z_i), i \le n) \cdot \mathbb{P}(Z_i = z_i, i \le n \mid Y_i = y_i, i \le n)\end{aligned}$$

By the Markov property for the two-coordinate chain,

$$\mathbb{P}(Y_{n+1} = y \mid (Y_i, Z_i) = (y_i, z_i), i \le n) = \mathbb{P}(Y_1 = y \mid (Y_0, Z_0) = (y_n, z_n)))$$

If $y_n = z_n$ then then the pair moves like lazy SRW so the latter probability is just $p_{y_n y}$, the correct value for lazy SRW. If $y_n \ne z_n$ then the latter probability is (a) $1/2$ if $y = y_n$, since then we need the Y-coordinate not to move ; (b) $1/2$ if $y = 0$, $y_n \ge 0$, since in this case we just need the Y coordinate to move ; (c) $1/2n$ if $y \ne 0$, $y_n = 0$, since in this case we need the y-coordinate not just to move but to move to the correct leaf. In all cases, the value is indeed $p_{y_n y}$. It follows that

$$\mathbb{P}(Y_{n+1} = y \mid Y_i = y_i, i \le n) = \sum_{z_0,\ldots,z_n} p_{y_n y} \cdot \mathbb{P}(Z_i = z_i, i \le n \mid Y_i = y_i, i \le n) = p_{y_n y}\,,$$

as required.

1 points (e) Let $\tau = \min(n : Y_n = Z_n)$. Show that for all $i, j \in \{0, 1, \ldots, s\}$, it holds that $\mathbb{P}_{(i,j)}(\tau \le 2) \ge 1/2$.

Answer. If $i = j$ this is obvious. If $i = 0$ and $j \ne 0$ then to have $\tau \le 2$ it suffices that the second coordinate moves first, which happens with probability $1/2$, so we in fact have $\mathbb{P}_{(i,j)}(\tau \le 1) \ge 1/2$. This stronger bound likewise holds if $i \ne 0$ and $j = 0$. Finally, if $i \ne 0$ and $j \ne 0$ then in the pair (Y_1, Z_1), exactly one coordinate equals zero, so we can use the Markov property and apply previous the bound for the probability $\tau \le 1$ to conclude.

1 points) (f) Show that the mixing time of lazy simple random walk on the star with s leaves is at most 4.

Answer. For all i and j,

$$\begin{aligned}
\mathbb{P}_{i,j}(\tau > 4) &= \sum_{k \neq l} \mathbb{P}_{i,j}(\tau \geq 4, (Y_2, Z_2) = (k,l)) \\
&= \sum_{k \neq l} \mathbb{P}_{i,j}(\tau > 2 \mid (Y_2, Z_2) = (k,l)) \mathbb{P}_{i,j}((Y_2, Z_2) = (k,l)) \\
&= \sum_{k \neq l} \mathbb{P}_{k,l}(\tau > 2) \cdot \mathbb{P}_{i,j}((Y_2, Z_2) = (k,l)) \\
&\leq \frac{1}{2} \sum_{k \neq l} \mathbb{P}_{i,j}((Y_2, Z_2) = (k,l)) \\
&\leq \frac{1}{2} \mathbb{P}_{i,j}(\tau > 2) \\
&\leq 1/4
\end{aligned}$$

Since each coordinate behaves like simple random walk, if $(Y_0, Z_0) = (i,j)$ then (Y_4, Z_4) is a coupling of $\delta_i P^4$ and $\delta_j P^4$, so

$$\|\delta_i P^4 - \delta_j P^4\|_{\mathrm{TV}} \leq \mathbb{P}_{(i,j)}(Y_4 \neq Z_4) = \mathbb{P}_{(i,j)}(\tau > 4) \leq 1/4.$$

Since i and j were arbitrary it follows that $d(4) \leq \bar{d}(4) \leq 1/4$, so the mixing time is at most 4.

3. Consider a Markov chain with state space $V = \{0, 1, \ldots, s\}$ with the following properties : (i) $0 < p_{ii} < 1$ for all $i \in V$; (ii) $p_{ii} + p_{i0} = 1$ for all $i \in \{1, \ldots, s\}$. (Such a chain only makes non-trivial moves along edges of the graph in Figure 3).

2 points (a) Find the stationary distribution of such a chain.

2 points (b) Prove that all chains of this form are reversible.

Answer. First note that if $p_{0i} = 0$ for some i then i is a transient state so $\pi(i) = 0$. Let N be the set of states $1 \leq i \leq s$ with $p_{0i} > 0$. For $i \in N$ let $q_i = p_{0i} \cdot (1 - p_{i0})/p_{i0}$.

Provided the Markov chain starts at 0 or at a state i in N, its transition probabilities are precisely that of simple random walk on the undirected weighted graph with vertices $\{0\} \cup N$, edges $\{\{0,i\}, i \in N\} \cup \{\{i,i\}, i \in N\}$ and edge weights $w(\{0,i\}) = p_{0i}$ and $w(\{i,i\}) = q_i$.

The standard formula for the stationary distribution of reversible Markov chains, plus the fact that $p_{0i} + q_i = p_{0i}/p_{i0}$ and $\sum_{i \in N} p_{0i} = 1$ gives that

$$\pi(i) = \frac{p_{0i}/p_{i0}}{1 + \sum_{i \in N}(p_{0i}/p_{i0})}, \ i \in N \quad \text{and} \quad \pi(0) = \frac{1}{1 + \sum_{i \in N}(p_{0i}/p_{i0})}.$$

Reversibility is immediate from the fact that† (after throwing away transient states) this chain is equivalent to random walk on a weighted graph.