Winter 2018 – Math 547 – MIDTERM 1 – February 8, 8 :35-9 :55

Read questions **carefully** before answering. State any result you use from class.

There are 18 points possible, but the exam will be marked out of 15, so it is possible to get $> 100\%$.

(**2 points**) 1. Let X be Poisson(λ)-distributed. Show that $\mathbb{E}(X(X-1)\dots(X-k+1)) = \lambda^k$ for all integers $k \geq 1$.

Answer. Note that the ordinary generating function of X is

$$G(s) = \mathbb{E}[s^X] = \sum_{n\geq 0} s^n \frac{\lambda^n e^{-\lambda}}{n!} = \exp(\lambda(s-1)).$$

It follows that $\mathbb{E}((X)_k) = G^{(k)}(0) = \lambda^k$.

2. Fix a sequence $(a_n, n \geq 0)$ of positive numbers, and let $F(s)$ be the ordinary generating function of the sequence. Express simply, in terms of $F(s)$, the ordinary generating functions of the following sequences :

(**1 point**) (a) $(a_n + 10, n \geq 0)$, where c is a constant.

(**1 point**) (b) $((n^2 + 2n + 1) \cdot a_n, n \geq 0)$

(**1 points**) (c) $(0, 0, 1, a_3, a_4, a_5, \dots)$

Answer.

For (a), the answer is

$$\sum_{n\geq 0}(a_n + 10)s^n = F(s) + \sum_{n\geq 0} 10s^n = F(s) + \frac{10}{1-s}\,.$$

For (b), use that $sF'(s) = \sum_{n\geq 0} na_n s^n$, and $s^2F''(n) = \sum_{n\geq 0} n(n-1)a_n s^n$. Thus the answer is

$$\sum_{n\geq 0}(n^2 + 2n + 1)a_n s^n = s^2F''(s) + 3sF'(s) + F(s).$$

For (c), use that $F(0) = a_0$, $sF'(0) = a_1 s$, $(s^2/2)F''(0) = a_2$. This gives

$$s^2 + \sum_{n\geq 3} a_n s^n = F(s) - F(0) - sF'(0) - \frac{s^2}{2}(F''(0) - 2).$$

(**2 points**) 3. Consider a sequence $(X_r, r \geq 1)$ of IID random variables. View these as the waiting times (in days) between events in a recurrent event process. (So that events occur at times $0, X_1, X_1 + X_2$, et cetera.)

Let L be the first instant that at least 10 days have passed since the most recent event. (So if $X_1 \geq 10$ then $L = 10$, for example.)

Show that

$$\mathbb{E}(s^L) = \frac{s^{10}\mathbb{P}(X_1 \geq 10)}{1 - \sum_{r=1}^{10} s^r \mathbb{P}(X_1 = r)}.$$

Answer. (There was a typo in the denominator, the sum should have ended at 9, not 10.) If $X_1 \geq 10$ then $L = 10$, and otherwise $L = X_1 + L'$, where L' is the first time *since* X_1 that at least ten days have passed since the most recent event. We can thus calculate

$$\begin{aligned}
\mathbb{E}(s^L) &= \sum_{i \geq 1} \mathbb{P}(X_1 = i)\mathbb{E}(s^L \mid X_1 = i) \\
&= s^{10}\mathbb{P}(X_1 \geq 10) + \sum_{i=1}^{9} \mathbb{P}(X_1 = i)\mathbb{E}(s^L \mid X_1 = i) \\
&= s^{10}\mathbb{P}(X_1 \geq 10) + \sum_{i=1}^{9} \mathbb{P}(X_1 = i)s^i\mathbb{E}(s^{L'} \mid X_1 = i) \\
&= s^{10}\mathbb{P}(X_1 \geq 10) + \sum_{i=1}^{9} \mathbb{P}(X_1 = i)s^i\mathbb{E}(s^L),
\end{aligned}$$

where in the last step we use the Markov property. Now solve for $\mathbb{E}(s^L)$.

4. A fair die is rolled repeatedly. Which of the following are Markov chains ? For those that are, supply the transition matrix.

(**1 point**) (a) The largest number X_n shown up to the nth roll.

(**1 point**) (b) The number N_n of sixes in n rolls.

(**1 point**) (c) At time r, the time C_r since the most recent six.

(**1 point**) (d) At time r, the time B_r until the next six.

Answer. These are all Markov chains. The transition probabilities p_{ij} are listed below.

For (a) the transition probabilities are

$$p_{ij} = \begin{cases} \frac{i}{6} & j = i \\ \frac{1}{6} & i < j \leq 6 \\ 0 & \text{otherwise.} \end{cases}$$

For (b) the transition probabilities are

$$p_{ij} = \begin{cases} \frac{1}{6} & j = i + 1 \\ \frac{5}{6} & j = i \\ 0 & \text{otherwise.} \end{cases}$$

For (c) the transition probabilities are

$$p_{ij} = \begin{cases} \frac{5}{6} & j = i + 1 \\ \frac{1}{6} & j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

For (d) the transition probabilities are

$$p_{ij} = \begin{cases} \left(\frac{5}{6}\right)^{j-1} \cdot \frac{1}{6} & 0 = i < j \\ 1 & j = i - 1 \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The trick in (d) is that even though it looks like the chain "needs to know the future", this "foreknowledge" can all be encapsulated within one step, which saves the Markov property.

5.

(**2 points**) (a) Classify, according to transience/recurrence and period, the states of the discrete-time Markov chains with state space $V = \{1, 2, 3, 4\}$ and transition matrices

$$(i) \quad \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (ii) \quad \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 0 & 2/3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Answer.

For (i), Once the chain reaches state 1 or 2 it never leaves. These two states intercommunicate, and both have self loops, so 1 and 2 are recurrent and have period 1. State 4 is also recurrent with period 1 since once 4 is reached the chain stays there forever. State 3 is transient, but has a self loop so also has period 1.

For (ii), The chain is recurrent as its graph has a cycle $4 \to 3 \to 1 \to 2 \to 4$ passing through all the states. Thus all states have the same period. It is only possible to return to 1, starting from 1, at even times, so the period is at least 2, and there is a cycle of length two starting from 1, so the period of the chain is 2.

(**2 points**) (b) For matrix (ii), work out $\mathbb{E}_3[\tau(4)]$.

Answer. Do a one-step calculation. From state 3 the chain always goes to 1 so

$$\mathbb{E}_3[\tau(4)] = 1 + \mathbb{E}_1[\tau(4)].$$

Another 1-step calculation yields that

$$\mathbb{E}_1[\tau(4)] = 1 + \frac{1}{2}\mathbb{E}_2[\tau(4)] + \frac{1}{2}\mathbb{E}_3[\tau(4)].$$

Using the previous identity for $\mathbb{E}_3[\tau(4)]$ this implies that

$$\mathbb{E}_1[\tau(4)] = 3 + \mathbb{E}_2[\tau(4)].$$

From 2 we return to 1 with probability $1/3$ and otherwise go to 4, so a further 1-step calculation gives

$$\mathbb{E}_2[\tau(4)] = 1 + \frac{1}{3}\mathbb{E}_1[\tau(4)] = 1 + \frac{1}{3}(3 + \mathbb{E}_2[\tau(4)]).$$

This gives $\mathbb{E}_2[\tau(4)] = 3$, so $\mathbb{E}_1[\tau(4)] = 6$ and thus $\mathbb{E}_3[\tau(4)] = 7$.

In general, writing down the one-step calculations from all nodes will give a system of linear equations which may be solved to find the answer.

(**2 points**) 6. Let i and j be states of a Markov chain. What does it mean mathematically to write that $i \leftrightarrow j$? Prove that if $i \leftrightarrow j$ then i and j have the same period.

Answer. Done in class and available in the class notes.