is an open rectangle. There are corresponding results for $\mathbb{R}^d$ and for more complicated geometries, but we shall not pursue them because they are unnecessary for the approach we take to the existence problem in the next chapter. Rather similar questions will however become urgent in Section 3.4, in a slightly different context.

In Section 2.1 a Poisson process will be formally defined as a random countable subset Π of S for which the random variables $N(A)$ counting the number of points of Π in the test sets A have two properties. One is the independence property described in Section 1.1. The other is that $N(A)$ has the Poisson distribution $\mathscr{P}(\mu)$, where the parameter μ depends on A. The dependence of μ on A is not arbitrary, and in fact μ must be a non-atomic measure on S.

Conversely, suppose that μ is a non-atomic measure on S. Can we be sure that there is a Poisson process for which $N(A)$ has distribution $\mathscr{P}(\mu(A))$ for all A? General theorems on the existence of random processes show that there are families of random variables $N(A)$ with the right joint distributions, but it is much more difficult (Kendall 1974) to show that these come from a random set Π through (1.18).

It is much easier to give an explicit construction, and this is done in Section 2.5 under very weak conditions on S and μ. These conditions are easily satisfied in all known applications, and the result is a theory of great generality and wide applicability. Before describing this, however, we show in the next section that it is only the Poisson distribution which can possibly give rise to such a simple theory.

1.4 The inevitability of the Poisson distribution

Think of a test set A_t which is in two dimensions a circle of radius t (though it could equally be a square of side t). Write

$$p_n(t) = \mathbb{P}\{N(A_t) = n\}, \tag{1.21}$$

$$q_n(t) = \mathbb{P}\{N(A_t) \leqslant n\}. \tag{1.22}$$

Because $N(A_t)$ increases with t, q_n decreases, so that q_n and p_n have only jump discontinuities and are differentiable almost everywhere. We give a heuristic derivation of a differential equation for p_n.

The random variable $N(A_t)$ jumps from n to $n + 1$ when its enlarging boundary crosses one of the random points. The probability that this occurs between t and $t + h$ is the probability that there is a point in the ring between A_t and A_{t+h}, which we suppose small when h is small. If we ignore the probability of two or more points in this ring, this probability (given the number in A_t) is the same as $\mu(t + h) - \mu(t)$, where

$$\mu(t) = \mathbb{E}\{N(A_t)\}. \tag{1.23}$$

If this is independent of the number of points in A_t, then for small h,

$$q_n(t) - q_n(t+h) = p_n(t)\{\mu(t+h) - \mu(t)\}.$$

Letting $h \to 0$,

$$-\frac{\mathrm{d}q_n}{\mathrm{d}t} = p_n \frac{\mathrm{d}\mu}{\mathrm{d}t}. \tag{1.24}$$

Since $p_n = q_n - q_{n-1}$, this means that

$$\frac{\mathrm{d}p_0}{\mathrm{d}t} = -p_0 \frac{\mathrm{d}\mu}{\mathrm{d}t}, \qquad \frac{\mathrm{d}p_n}{\mathrm{d}t} = (p_{n-1} - p_n)\frac{\mathrm{d}\mu}{\mathrm{d}t} \quad (n \geqslant 1). \tag{1.25}$$

The first of these equations shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log p_0 + \mu) = 0$$

and since $p_0(0) = 1$, $\mu(0) = 0$,

$$\log p_0 + \mu = 0$$

so that

$$p_0(t) = \mathrm{e}^{-\mu(t)} \tag{1.26}$$

for all t. The second equation in (1.25) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}(p_n \,\mathrm{e}^{\mu}) = p_{n-1}\,\mathrm{e}^{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}t}$$

so that, since $p_n(0) = 0$ $(n \geqslant 1)$,

$$p_n(t) = \mathrm{e}^{-\mu(t)} \int_0^t p_{n-1}(s)\,\mathrm{e}^{\mu(s)} \frac{\mathrm{d}\mu}{\mathrm{d}s}\,\mathrm{d}s. \tag{1.27}$$

From this it follows by induction on n, starting at (1.26), that

$$p_n(t) = \mathrm{e}^{-\mu(t)}\,\mu(t)^n/n! \tag{1.28}$$

so that $N(A_t)$ has distribution $\mathscr{P}(\mu(t))$.

This argument has clearly relied on several implicit assumptions, but it does suggest that the Poisson distribution is an inevitable consequence of the 'complete randomness' inherent in our assumptions of independence. We shall return to this more systematically in Chapter 8.