6.13 Spatial Poisson Processes

Basic setting: a (measurable) set $D \subset \mathbb{R}^d$. (Imagine $D = (0,1)^d$ or $D = \mathbb{R}^d$)

(Much extends to more general measurable spaces.)

Let $\mathcal{B} = \{\text{Measurable sets } M \subset D\}$. [Show Banach-Tarski paradox video.]

Fix a fn $\lambda : D \to [0, \infty)$ which is nice* (think bounded, continuous; return to this below)

For $M \subset D$ measurable, write $\Lambda(M) := \int_M \lambda(\vec{x})\, d\vec{x} = \int\cdots\int_{(x_1,\ldots,x_n)\in M} \lambda(x_1,\ldots,x_n)\, dx_1 \cdots dx_n$. (so $\Lambda : \mathcal{B} \to [0,\infty)$)

Def: A Poisson point process with rate (or intensity) function λ is a random set $P \subset D$ s.t.

1. for all $M \subset D$ measurable, $|P \cap M| \overset{d}{=} \text{Poisson}(\Lambda(M))$
2. If $A_1, \ldots, A_n \subset D$ are disjoint and measurable then $|P \cap A_1|, \ldots, |P \cap A_n|$ are indep.

Λ is called the mean measure of P

If P is a Poisson point process with intensity fn λ / mean measure Λ write $P \sim PPP(\Lambda)$ or $P \sim PPP(\lambda)$

NB: If $X \sim Po(x)$, $Y \sim Po(y)$ are indep. then $X + Y \sim Po(x+y)$.

So $|P \cap A| \sim Po(\Lambda(A))$, $|P \cap B| \sim Po(\Lambda(B))$, and $P \cap A$ indep. of $P \cap B \Rightarrow |P \cap (A \cup B)| \sim Po(\Lambda(A) + \Lambda(B)) = Po(\Lambda(A \cup B))$.

nice: In fact just need a σ-finite measure $\Lambda: \mathcal{B} \to [0,\infty)$ with the bisection property

σ-finite: Can find $(D_i, i \geq 1)$, $D_i \in \mathcal{B}$, $\bigcup_{i\geq 1} D_i = D$, $\Lambda(D_i) < \infty$ for all i

bisection property: for all $M \in \mathcal{B}$ there are $M_1, M_2 \in \mathcal{B}$ s.t. $M_1 \cup M_2 = M$, $M_1 \cap M_2 = \emptyset$, $\Lambda(M_1) = \Lambda(M_2) = \frac{\Lambda(M)}{2}$.

Exercise: If $\lambda: \mathbb{R}^d \to (0,\infty)$ bounded and $\Lambda(M) = \int_M \lambda(x)\,dx$ then Λ is nice.

The following is a key use of niceness.

Disjointness Lemma Suppose P and P' are indep. Poisson processes with mean measures Λ and Λ'. Suppose Λ and Λ' both have the bisection property.

If $\Lambda(D) < \infty$ and $\Lambda'(D) < \infty$ then $\mathbb{P}(P \cap P' \neq \emptyset) = 0$.

Proof Idea: If $M \subset D$ and $\Lambda(M) < \varepsilon$, $\Lambda'(M) < \varepsilon$ then $\mathbb{P}(P \cap P' \cap M \neq \emptyset) \leq \mathbb{P}(|P \cap M| \geq 1, |P' \cap M| \geq 1)$
$\leq \mathbb{P}(|P \cap M| \geq 1) \cdot \mathbb{P}(|P' \cap M| \geq 1)$
$\leq \mathbb{E}|P \cap M| \cdot \mathbb{E}|P' \cap M| < \varepsilon^2$.

Key: Quadratic in ε.

Fix $m \in \mathbb{N}$, let $n = 2^m$. Bisect D m times to form $2^m = n$ disjoint pieces $D_1, \ldots, D_n$ with $\Lambda(D_1) = \cdots = \Lambda(D_n) = \Lambda(D)/n$ and $\bigcup_{i\leq n} D_i = D$

Now for each $i \leq n$, bisect D_i m times to form $2^m = n$ disjoint pieces $D_{i1}, \ldots, D_{in}$ with $\Lambda'(D_{i1}) = \cdots = \Lambda'(D_{in}) = \Lambda'(D_i)/n$ and $\bigcup_{j\leq n} D_{ij} = D_i$

Let $E_{ij} = \{|P \cap D_{ij}| \geq 1, |P' \cap D_{ij}| \geq 1\}$.

If $P \cap P' \neq \phi$ then there is $i \leq n$, $j \leq n$ such that E_{ij} occurs.

So $\mathbb{P}(P \cap P' \neq \phi) \leq \sum_{i \leq n} \sum_{j \leq n} \mathbb{P}(E_{ij})$.

Next, $\mathbb{P}(E_{ij}) = \mathbb{P}(|P \cap D_{ij}| \geq 1) \cdot \mathbb{P}(|P' \cap D_{ij}| \geq 1)$ $\quad \mathbb{P}(X \geq 1) \leq \mathbb{E}X$: Markov

$\leq \mathbb{E}(|P \cap D_{ij}|) \cdot \mathbb{E}(|P' \cap D_{ij}|)$

$= \Lambda(D_{ij}) \cdot \Lambda'(D_{ij})$

Thus $\mathbb{P}(P \cap P' \neq \phi) \leq \sum_{i \leq n} \sum_{j \leq n} \Lambda(D_{ij}) \Lambda'(D_{ij})$

$= \sum_{i \leq n} \sum_{j \leq n} \Lambda(D_{ij}) \cdot \Lambda'(D_i)/n$

$= \sum_{i \leq n} \Lambda'(D_i)/n \cdot \sum_{j \leq n} \Lambda(D_{ij})$

$= \sum_{i \leq n} \Lambda'(D_i)/n \cdot \Lambda(D_i) = \sum_{i \leq n} \Lambda'(D_i)/n \cdot \Lambda(D)/n = \dfrac{\Lambda'(D)\Lambda(D)}{n^2}$.

But $n = 2^m$ and $m \geq 1$ is arbitrary. so $\mathbb{P}(P \cap P' \neq \phi) \leq \inf_{m \geq 0} \dfrac{\Lambda'(D)\Lambda(D)}{2^{2m}} = 0$. $\square$

Superposition Theorem Suppose $P \sim PPP(\Lambda)$, $P' \sim PPP(\Lambda')$, and P and P' are independent. Then $P + P' \sim PPP(\Lambda + \Lambda')$.

Proof of superposition theorem: (A) assume $\Lambda(D) < \infty$, $\Lambda'(D) < \infty$.

Key point: If $P \cap P' = \phi$ then for all $M \in \mathcal{B}$, $|(P \cup P') \cap M| = |P \cap M| + |P' \cap M|$

By disjointness lemma, $\mathbb{P}(P \cap P' = \phi) = 1$. NOTATION: $P \cap P' \overset{a.s.}{=} \phi$ So for all $M \in \mathcal{B}$,

$|(P \cup P') \cap M| \overset{d}{=} |P \cap M| + |P' \cap M| \overset{d}{=} Poi(\Lambda(M)) + Poi(\Lambda'(M)) = Poi(\Lambda(M) + \Lambda'(M))$. ✓ First point.

$= Poi\left(\int_M (\lambda(x) + \lambda'(x))\,dx\right)$

Also, for all disjoint $M_1, \ldots, M_k$, the random variables $(|P \cap M_i|, i \le k)$ and $(|P' \cap M_i|, i \le k)$ are indep. Thus so are the r.v.s $(|P \cap M_i| + |P' \cap M_i|, i \le k) \overset{a.s.}{=} (|(P \cup P') \cap M_i|, i \le k)$. Second point. ✓

(B) If $\Lambda(D) = \infty$ or $\Lambda'(D) = \infty$, write $D = \bigcup_{i \ge 1} D_i$ where $\Lambda(D_i) < \infty$, $\Lambda'(D_i) < \infty$ for all i and D_i, $i \ge 1$ are disjoint. This is possible since Λ, Λ' are σ-finite.

Write $P_i = P \cap D_i$, $P_i' = P' \cap D_i$ for the restrictions of P_i and P_i' to D_i.

$\Lambda_i(M) = \Lambda(M \cap D_i)$ $\Lambda_i'(M) = \Lambda_i'(M \cap D_i)$.

Note that P_i and P_i' are indep. Poisson processes with mean measures Λ_i, Λ_i'.

Using part Ⓐ, follows that $P_i \cup P_i'$ is a Poisson process with mean measure $\Lambda_i + \Lambda_i'$.

Now if $M \subset D$ then

$$|M \cap (P \cup P')| = \sum_{i \geq 1} |M \cap (P_i \cup P_i')| \overset{\text{a.s.}}{=} \sum_{i \geq 1} |M \cap P_i| + \sum_{i \geq 1} |M \cap P_i'| \overset{d}{=} \text{Po}(\Lambda(M \cap D) + \Lambda'(M \cap D))$$

$\text{Po}(\sum_{i \geq 1} \Lambda(M \cap D_i)) = \text{Po}(\Lambda(M \cap D))$

$\text{Po}(\sum_{i \geq 1} \Lambda'(M \cap D_i)) = \text{Po}(\Lambda'(M \cap D))$

Last equality because P and P' are indep.

And if $(M_i, i \geq 1)$ disjoint then $(|(P \cup P') \cap M_i|, i \geq 1)$ are indep. just as before. □

Mapping Theorem Fix measurable sets D, D' (could have $D \subseteq \mathbb{R}^d$, $D' \subseteq \mathbb{R}^k$).
Write $\mathcal{B} = \{\text{meas. sets in } D\}$, $\mathcal{B}' = \{\text{meas. sets in } D'\}$.

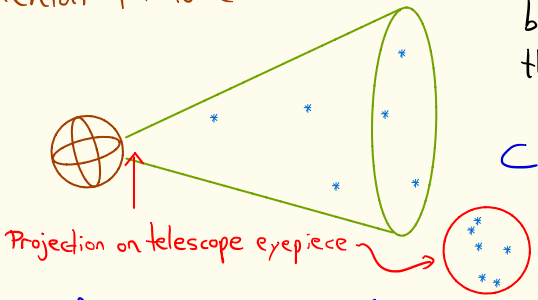
Let Λ be a measure on D and $f: D \to D'$ measurable.

Let $P \sim \text{PPP}(\Lambda)$, and define a measure Λ^* on $\mathcal{B}'$ by setting

$\Lambda^*(N) := \Lambda(f^{-1}(N))$, where $f^{-1}(N) = \{x \in D : f(x) \in N\}$.

If Λ^* has the bisection property then $P := f(P) = \{f(p), p \in P\}$ is $\text{PPP}(\Lambda^*)$.

Mental Picture:



Proof Assume $\Lambda(D)$ finite. (If Λ is σ-finite but not finite then cut into many pieces as in superposition theorem.)

Claim: $\mathbb{P}(\exists\, p,p' \in P \text{ s.t. } f(p)=f(p')) = 0$.

NB This may "feel obvious" but note no need for f to be continuous or injective! (Think gravitational lensing.)

Proof: Fix $m \geq 1$, bisect D^* m times to obtain $n = 2^m$ disjoint sets $D_1^*, \ldots, D_n^*$ with $\cup D_i^* = D^*$ and $\Lambda^*(D_1^*) = \cdots = \Lambda^*(D_n^*) = \Lambda^*(D^*)/n = \Lambda(D)/n$. With $D_i = f^{-1}(D_i^*)$, $i \leq n$, then $\Lambda(D_i) = \Lambda^*(D_i^*) = \Lambda(D)/n$ for all n. Now, if there are $p,p' \in P$ with $f(p)=f(p')$ then there is $i \leq n$ s.t. $f(p) \in D_i^*$, $f(p') \in D_i^*$, so there is $i \leq n$ (the same i!) s.t. $p \in D_i$, $p' \in D_i$.

So $\mathbb{P}(\exists\, p,p' \in P \text{ s.t. } f(p)=f(p')) \leq \mathbb{P}(\exists\, i \leq n \text{ s.t. } |P \cap D_i| \geq 2) \leq n \cdot \mathbb{P}(\text{Poisson}(\Lambda(D)/n) \geq 2)$

$\mathbb{P}(\text{Poisson}(\mu) \geq 2) = 1 - e^{-\mu} - \mu e^{-\mu} = 1 - (1+\mu)e^{-\mu}$ $\qquad e^{-\mu} \geq 1-\mu$

$\leq 1 - (1+\mu)(1-\mu) = \mu^2$.

So $\mathbb{P}(\exists\, p,p' \in P \text{ s.t. } f(p)=f(p')) \leq n \cdot (\Lambda(D)/n)^2 = \Lambda(D)/n$. But $n = 2^m$ and $m \geq 1$ arbitrary; so this prob. $= 0$. ∎

Now: ① For $N \in \mathcal{B}'$, $|P' \cap N| \overset{a.s.}{=} |\{p \in P : f(p) \in N\}| = |\{p \in P : p \in f^{-1}(N)\}| = |P \cap f^{-1}(N)|$

But $|P \cap f^{-1}(N)| \overset{d}{=} \text{Poisson}(\Lambda(f^{-1}(N))) = \text{Poisson}(\Lambda^*(N))$ by def. of Λ^*.

② If $N_1, \dots, N_k \subset D$ disjoint then $f^{-1}(N_1), \dots, f^{-1}(N_k)$ disjoint

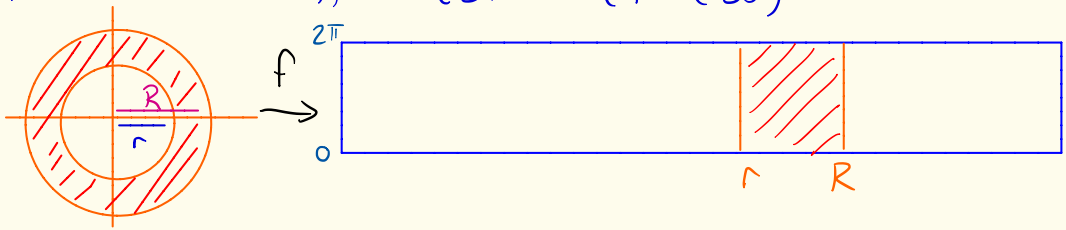
so $(|P' \cap N_i|, i \le k) \overset{a.s.}{=} (|P \cap f^{-1}(N_i)|, i \le k)$ are independent.

Thus $P' \sim PPP(\Lambda^*)$ ∎

Example: Polar coordinates $P \sim PPP(\Lambda)$ on $\mathbb{R}^2$, $\Lambda(B) = \int_B \lambda \, dt$ — constant.

$(x,y) = re^{i\Theta}$ $r \ge 0$, $\Theta \in [0, 2\pi)$ $\quad f(x,y) = (r, \Theta)$ $r = |(x,y)| = \sqrt{x^2+y^2}$ $\quad \Theta = \tan^{-1}(y/x)$

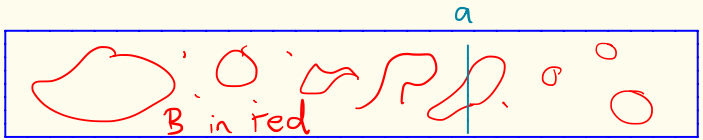
For $B \subset [0,\infty) \times [0, 2\pi)$, $\Lambda^*(B) = \Lambda(f^{-1}(B))$



Prop: Λ^* is nice.

Lemma: For all $B \subset [0,\infty)\times[0,2\pi)$, the function $\Lambda^*(B \cap [0,a]\times[0,2\pi))$ is continuous in a.

Proof of lemma: Fix $B \subset [0,\infty)\times(0,2\pi)$.



Then $\Lambda^*(B \cap [0,a]\times[0,2\pi))$ increasing in a.

So if Λ^* is not continuous then $\exists$ a s.t.

$$\Lambda^*(B\cap[0,a]\times[0,2\pi)) > \Lambda^*(B\cap[0,a)\times[0,2\pi))$$

This means $\Lambda'(\ell) > 0$, where $\ell = \{a\}\times[0,2\pi)$.

But $\Lambda'(\ell) = \Lambda(f^{-1}(\ell)) = \lambda \cdot \text{area}(f^{-1}(\ell)) = \lambda \cdot \text{area}(\{x\in\mathbb{R}^2 : |x| = a\}) = 0$. So continuous. ▨

Proof of prop: Now know $\Lambda^*(B\cap[0,a]\times[0,2\pi))$ increases continuously from 0 to 1 as a increases from 0 to ∞. So there is a^* s.t. $\Lambda^*(B\cap[0,a^*]\times[0,2\pi)) = \Lambda^*(B)/2$ □

By mapping theorem, follows that $f(P)$ is $PPP(\Lambda^*)$, i.e. has rate function $\lambda^*((r,\theta)) = \lambda \cdot r$.

Instead of "bisection", a more classical "niceness" condition is that $\Lambda(\{x\}) = 0 \ \forall x \in D$.
In general measurable spaces $(D, \mathcal{B})$ (not necessarily $\subset \mathbb{R}^d$) then need to assume that $\{x\}^c \in \mathcal{B}$ as well! (As in book)

In fact "standard arguments" to construct Poisson processes on general measurable spaces assume that

the diagonal $\Delta := \{(x,x), x \in D\}$ is product measurable, before completion, and that $\Lambda \times \Lambda(\Delta) = 0$.

Theorem: Let μ be a σ-finite measure on a measurable space $(D, \mathcal{B})$. Then
μ is nice $\iff (\mu \times \mu)^*(\Delta) = 0$, where $(\mu \times \mu)^*$ is the product outer measure.

In particular, if Δ is a measurable subset of $S \times S$ and μ is non-atomic then μ is nice.

The Poisson process is "natural"

Idea: Independence plus "not too many points" defines the Poisson process.

Fix a nice mean measure Λ on $\mathbb{R}^d$; Let $P \subset \mathbb{R}^d$ be some random set with $\mathbb{E}\,|P \cap A| = \Lambda(A)$ for measurable Λ and satisfying the independence assumption for Poisson processes.

Write $N_t = |\{p \in P : |p| \le t\}|$, $\left(|p| = \sqrt{p_1^2 + \cdots + p_d^2}\right)$ $\mu(t) = \mathbb{E}\,N_t$

Let $p_n(t) = \mathbb{P}(N_t = n)$, $q_n(t) = \mathbb{P}(N_t \le n)$

Note $q_n(t) - q_n(t+h) \le \mathbb{P}(N_t = n,\ \exists \text{ a point } p \in P \text{ with } t < |p| \le t+h)$ ← dodgy because ignores possibility of 2 points in annulus

$= \mathbb{P}(N_t = n)\,\mathbb{P}(\exists \text{ a point } p \in P \text{ with } t < |p| \le t+h)$

$= (1+o(1))\,\mathbb{P}(N_t = n)\,\mathbb{E}(\#\text{ points } p \in P \text{ with } t < |p| \le t+h)$ ← dodgy for basically the same reason.

$= (1+o(1))\,p_n(t)\,(\mu(t+h) - \mu(t))$

$q_n(t)$ decreasing

Now divide by h, let $h \to 0$ to get

$$q_n'(t) = -p_n(t)\cdot\mu'(t).$$

Next, $p_0(t) = q_0(t)$ so $p_0'(t) = -p_0(t)\mu'(t)$, (A)

and for $n \ge 1$, $p_n(t) = q_n(t) - q_{n-1}(t)$, so

$$p_n'(t) = q_n'(t) - q_{n-1}'(t) = (p_{n-1}(t) - p_n(t))\,\mu'(t). \quad (B)$$

Prop: If $(p_n(t), n \geq 0)$ and $\mu(t)$ satisfy (A) and (B) with boundary conditions $\mu(0)=0$, $p_0(0)=1$, $p_n(0)=0$, $n \geq 1$, then $p_n(t) = \frac{e^{-\mu(t)}\mu(t)^n}{n!}$ for all $n \in \mathbb{N}$, $t \geq 0$.

Proof Induction on n.

$n=0$ First, $\frac{d}{dt}\left(\log p_0(t) + \mu(t)\right) = \frac{p_0'(t)}{p(t)} + \mu'(t) \overset{(A)}{=} -\mu'(t) + \mu'(t) = 0.$

So $p_0(t) = \exp(\mu(0) - \mu(t)) = \exp(-\mu(t))$.

$n>0$ Next, $\frac{d}{dt}\left(p_n(t)e^{\mu(t)}\right) = p_n'(t)e^{\mu(t)} + p_n(t)\mu'(t)e^{\mu(t)}$

$$\overset{(B)}{=} \left(p_{n-1}(t) - p_n(t)\right)\mu'(t)e^{\mu(t)} + p_n(t)\mu'(t)e^{\mu(t)}$$

$$= p_{n-1}(t)\mu'(t)e^{\mu(t)}$$

And $p_n(0)e^{\mu(0)} = 0$ (since $n \geq 1$, $p_n(0)=0$)

so integrating gives $p_n(t)e^{\mu(t)} = \int_0^t p_{n-1}(s)e^{\mu(s)}\mu'(s)\,ds$

$$\overset{\text{induction}}{=} \int_0^t \frac{e^{-\mu(s)}\mu(s)^{n-1}}{(n-1)!} e^{\mu(s)}\mu'(s)\,ds\,ds$$

$$= \int_0^t \frac{\mu(s)^{n-1}}{(n-1)!}\mu'(s)\,ds \overset{\text{integration by parts}}{=} \frac{\mu(t)^n}{n!},$$ which completes induction. □

Conditioning on # of points

Setup: P Poisson process on D, nice mean measure Λ.

Prop

Given $A \subset D$ with $\Lambda(A) < \infty$, conditional dist. of $P \cap A$ given $|P \cap A| = n$ is same as n IID points $X_1, \dots, X_n$, each with dist. given by $\mathbb{P}(X_i \in B) = \frac{\Lambda(B)}{\Lambda(A)}$ for $B \subset A$ measurable.

Proof

If $B_1, \dots, B_k$ partition A

$$\mathbb{P}(|B_1 \cap P| = n_1, \dots, |B_k \cap P| = n_k) = \mathbb{P}(|B_1 \cap P| = n_1) \cdot \dots \cdot \mathbb{P}(|B_k \cap P| = n_k)$$
$$= e^{-\Lambda(B_1)} \frac{\Lambda(B_1)^{n_1}}{n_1!} \cdot \dots \cdot e^{-\Lambda(B_k)} \frac{\Lambda(B_k)^{n_k}}{n_k!}$$

So if $n_1, \dots, n_k = n$ then

$$\mathbb{P}(|B_1 \cap P| = n_1, \dots, |B_k \cap P| = n_k \mid |A \cap P| = n) = \frac{\mathbb{P}(|B_1 \cap P| = n_1, \dots, |B_k \cap P| = n_k, |A \cap P| = n)}{\mathbb{P}(|A \cap P| = n)}$$

$$= \frac{n!}{n_1! \cdots n_k!} \left(\frac{\Lambda(B_1)}{\Lambda(B)}\right)^{n_1} \cdots \left(\frac{\Lambda(B_k)}{\Lambda(B)}\right)^{n_k}.$$

$\mathbb{P}(|A \cap P| = n) = e^{-\Lambda(A)} \frac{\Lambda(A)^n}{n!}$

Likewise, if $X_1, \dots, X_n$ are IID as above and $X = \{X_1, \dots, X_k\}$ then

$$\mathbb{P}(|B_1 \cap X| = n_1, \dots | B_k \cap X| = n_k) = \frac{n!}{n_1! \cdots n_k!} \cdot \left(\frac{\Lambda(B_1)}{\Lambda(B)}\right)^{n_1} \cdots \left(\frac{\Lambda(B_k)}{\Lambda(B)}\right)^{n_k}. \quad \square$$

Colouring theorem

Setup: P Poisson process on D, intensity fn λ. $\Lambda(B) = \int_B \lambda(\vec{x})\, d\vec{x}$. Colouring fn $\beta : D \to [0,1]$.

Independently for each point $p \in P$, colour p blue with probability $\beta(p)$

otherwise colour p green (so green with probability $\gamma(p) = 1 - \beta(p)$)

Let $G = \{\text{green points in } P\}$, $B = \{\text{blue points in } P\}$.

Then B is a P.P.P. with intensity fn $\lambda(x)\beta(x)$, G is a PPP with intensity fn $\lambda(x)\gamma(x)$, and B, G independent.

Proof:

Fix disjoint $A_1, \ldots, A_n \subset D$ with $\Lambda(A_i) < \infty$ for all i.

Must prove: $|A_1 \cap B|, |A_1 \cap G|, \ldots, |A_n \cap B|, |A_n \cap G|$ independent,

and $|A_i \cap B| \sim \text{Poi}\left(\int_{A_i} \lambda(x)\beta(x)\,dx\right)$, $|A_i \cap G| \sim \text{Poi}\left(\int_{A_i} \lambda(x)\gamma(x)\,dx\right)$

I.e. $$\mathbb{P}(|A_i \cap B| = b_i, |A_i \cap G| = g_i, 1 \le i \le n) = \prod_{i=1}^{n} \mathbb{P}\left(\text{Poi}\left(\int_{A_i} \lambda(x)\beta(x)\,dx\right) = b_i\right) \cdot \prod_{i=1}^{n} \mathbb{P}\left(\text{Poi}\left(\int_{A_i} \lambda(x)\gamma(x)\,dx\right) = g_i\right)$$

Step 1: Disjoint regions independent so

$$\mathbb{P}(|A_i \cap B| = b_i, |A_i \cap G| = g_i, 1 \le i \le n) = \prod_{i=1}^{n} \mathbb{P}\left(\text{Poi}\left(\int_{A_i} \lambda(x)\beta(x)\,dx\right) = b_i, \text{Poi}\left(\int_{A_i} \lambda(x)\gamma(x)\,dx\right) = g_i\right)$$

So can just focus on a single region $A = A_i$

Step 2 Fix a region $A \subset D$ with $\Lambda(A) < \infty$ and $b, g \in \mathbb{N}$. Let $m = b+g$.

Calculate $\mathbb{P}(|B \cap A| = b,\ |G \cap A| = g) = \mathbb{P}(|B \cap A| = b,\ |G \cap A| = g,\ |P \cap A| = m)$

$= \underbrace{\mathbb{P}(|P \cap A| = n)}_{①} \cdot \underbrace{\mathbb{P}(|B \cap A| = b,\ |G \cap A| = g \mid |P \cap A| = m)}_{②}$

① $= \mathbb{P}(\mathrm{Poi}(\Lambda(A)) = m) = \dfrac{e^{-\Lambda(A)} \Lambda(A)^m}{m!}$.

For ②: given that $|P \cap A| = m$, the points $P \cap A$ are indep. with dist. as in conditioning result.

A single random point in A will have $\mathbb{P}(X \text{ is blue}) = \int_A \mathbb{P}(X \text{ is blue} \mid X = x) \cdot \dfrac{\lambda(x)}{\Lambda(A)}\, dx = \int_A \beta(x) \dfrac{\lambda(x)}{\Lambda(A)}\, dx =: \bar{\beta}$

$\mathbb{P}(X \text{ is green}) = \int_A \mathbb{P}(X \text{ is green} \mid X = x) \cdot \dfrac{\lambda(x)}{\Lambda(A)}\, dx = \int_A \gamma(x) \dfrac{\lambda(x)}{\Lambda(A)}\, dx = \bar{\gamma} = 1 - \bar{\beta}$

So ② $= \mathbb{P}(\mathrm{Bin}(m, \bar{\beta}) = b) = \binom{m}{b} \bar{\beta}^b \bar{\gamma}^g = \dfrac{m!}{b!g!} \bar{\beta}^b \bar{\gamma}^g$

Get $\mathbb{P}(|B \cap A| = b,\ |G \cap A| = g) = \dfrac{e^{-\Lambda(A)} \Lambda(A)^m}{m!} \cdot \dfrac{m!}{b!g!} \bar{\beta}^b \bar{\gamma}^g = e^{-\bar{\beta}\Lambda(A)} \dfrac{\bar{\beta}^b}{b!} \cdot e^{-\bar{\gamma}\Lambda(A)} \dfrac{\bar{\gamma}^g}{g!} = \circledast$

Finally, $\bar{\beta}\Lambda(A) = \Lambda(A) \int_A \beta(x) \dfrac{\lambda(x)}{\Lambda(A)}\, dx = \int_A \beta(x) \lambda(x)\, dx$ $\qquad$ $\bar{\gamma}\Lambda(A) = \Lambda(A) \int_A \gamma(x) \dfrac{\lambda(x)}{\Lambda(A)}\, dx = \int_A \gamma(x) \lambda(x)\, dx$

So $\circledast = \exp\left(-\int_A \beta(x)\lambda(x)\, dx\right) \dfrac{\bar{\beta}^b}{b!} \cdot \exp\left(-\int_A \gamma(x)\lambda(x)\, dx\right) \dfrac{\bar{\gamma}^g}{g!} = \mathbb{P}\left(\mathrm{Poi}\left(\int_A \beta(x)\lambda(x)\, dx\right) = b\right) \cdot \mathbb{P}\left(\mathrm{Poi}\left(\int_A \gamma(x)\lambda(x)\, dx\right) = g\right)$

Rényi's Theorem

Let Λ be a nice mean measure on D. Suppose P is a random subset of D s.t. for all $A \subset D$ measurable, a) $\mathbb{E}|P \cap A| = \Lambda(A)$; b) $\mathbb{P}(|P \cap A| = 0) = e^{-\Lambda(A)}$. Then $P \sim PPP(\Lambda)$.

Proof: **Step 1:** ("Empty events independent")

Fix $A_1, \dots, A_n$ disjoint $\subset D$.

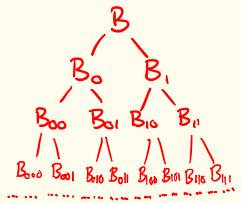
Then $$\mathbb{P}(|A_i \cap P| = 0,\ i \le n) = \mathbb{P}\left(\left|\left(\bigcup_{i \le n} A_i\right) \cap P\right| = 0\right)$$
$$= \exp\left(-\Lambda\left(\bigcup_{i \le n} A_i\right)\right) = \exp\left(-\sum_{i \le n} \Lambda(A_i)\right)$$
$$= \prod_{i \le n} \exp(-\Lambda(A_i)) = \prod_{i \le n} \mathbb{P}(|A_i \cap P| = 0),$$

So, the events $\{|A_i \cap P| = 0\}$, $1 \le i \le n$ are independent.

Step 2: (Poisson regions)

Fix $B \subset D$, $\Lambda(B) = x \in (0, \infty)$.

Bisect B to form $B = B_0 \cup B_1$, $\Lambda(B_0) = \Lambda(B_1) = \frac{x}{2}$.



Recursively continue. At level m have sets $(B_\sigma, \sigma \in \{0,1\}^m)$ partitioning B, all of measure $\frac{x}{2^m}$.

For $m \ge 0$, $\sigma \in \{0,1\}^m$ let E_σ be the event $\{|B_\sigma \cap P| = \phi\}$

NB. The events $(E_\sigma, \sigma \in \{0,1\}^m)$ are indep. with prob. $\exp(-\frac{x}{2^m})$

Let $N_m(B) = \#$ of $\sigma \in \{0,1\}^m$ s.t. $|B_\sigma \cap P| \neq \phi$.

Clearly $N_m(B) \le N_{m+1}(B)$ so there is a limit random variable $N_\infty(B) = \lim_{n \to \infty} N_m(B)$.

Note $N_m(B) \le |P \cap B|$ for all m so $N_\infty(B) \le |P \cap B|$ also.

Note: $N_m(B) = \#\{\sigma \in \{0,1\}^m : |B_\sigma \cap P| \neq \phi\} \stackrel{d}{=} Bin(2^m, 1-\exp(-\frac{x}{2^m}))$

Recall: If $n(m) \to \infty$, $p(m) \to 0$ s.t. $n(m) \cdot p(m) \to \lambda$ then $Bin(n(m), p(m)) \xrightarrow{d} Poi(\lambda)$.
(Related: Exercise 5.2.4, assignment 1. See also example 3.5.4 of textbook.)

Since $2^m \cdot (1-\exp(-\frac{x}{2^m})) = 2^m\left(\frac{x}{2^m} + o\left(\frac{x}{2^m}\right)\right) \to x$ as $m \to \infty$, have $N_m(B) \xrightarrow{d} Poi(x)$.

So $N_\infty(B) \stackrel{d}{=} Poi(x)$.

Now, $N_\infty(B) \leq |P \cap B|$ but $\mathbb{E}N_\infty(B) = x = \mathbb{E}|P \cap B|$.

So $|P \cap B| - N_\infty(B) \geq 0$, $\mathbb{E}(|P \cap B| - N_\infty(B)) = 0$.

So $\mathbb{P}(N_\infty(B) = |P \cap B|) = 1$, so $|P \cap B| \stackrel{d}{=} Poi(x)$.

Step 3 Full Independence

We prove pairwise independence to keep notation simple. Mutual independence no harder.

Key tool: If $X_m \xrightarrow{a.s.} X$, $Y_m \xrightarrow{a.s.} Y$ and X_m, Y_m indep. for all m then X, Y indep.
(overkill: only need convergence in distribution. But a.s. conv $\Rightarrow$ conv. in dist.)

Because then $\mathbb{P}(X \leq x, Y \leq y) = \lim_{m\to\infty} \mathbb{P}(X_m \leq x, Y_m \leq y) = \lim_{m\to\infty} \mathbb{P}(X_m \leq x) \cdot \mathbb{P}(Y_m \leq y) = \mathbb{P}(X \leq x) \cdot \mathbb{P}(Y \leq y)$.

Fix disjoint regions $A, B \subset D$. Do repeated bisection of both regions, as in step 2.

Let $N_m(A) = \#\{\sigma \in \{0,1\}^m : |A_\sigma \cap P| \neq \phi\}$

$N_m(B) = \#\{\sigma \in \{0,1\}^m : |B_\sigma \cap P| \neq \phi\}$

The events $|A_\sigma \cap P| \neq \phi$, $|B_\sigma \cap P| = \phi$ are all independent by Step 1.

So $N_m(A)$, $N_m(B)$ independent.

By Step 2, $N_m(A) \xrightarrow{a.s.} |P \cap A|$, $N_m(B) \xrightarrow{a.s.} |P \cap B|$. So $|P \cap A|$, $|P \cap B|$ independent as required.