Math 547 – Assignment 5

Assigned on April 2, 2018. Due on April 16 at 11 PM, via myCourses.

1 Exercises

Feel free to discuss with others, but write your own solutions in your own words. Explain your work.
The first four questions are from the book.

1. 6.13. 1
2. 6.13.3
3. 6.13.7
4. 6.13.8
5. Let P be a Poisson process on $[0, \infty)$ with rate function $\lambda(t) = c/(1+t)$. Let X be the point of P closest to 0. Show that $\mathbb{E}(X) < \infty$ if and only if $c > 1$.
6. An apartment is listed for sale, and purchase offers begin to arrive. The offers are independent and identically distributed, and have values $(X_i, i \geq 1)$. Assume the X_i have density f and cumulative distribution function F.

   (a) Write $Y_1 = X_1$, and for $n \geq 1$ let Y_{n+1} be the value of the first offer exceeding Y_n. Show that $\{Y_i, i \geq 1\}$ is a Poisson process with intensity function $\lambda(t) = f(t)/(1 - F(t))$.

   (b) Let $Z_1 = X_k$, where X_k is the first "penultimate" offer. In other words, X_k is the *second* largest among $\{X_1, \ldots, X_k\}$, and k is the first value for which this occurs. Let Z_2 be the second "penultimate" offer, and so on.
   For example, if $(X_i, i \geq 1) = (2, 7, 1, 5, 10, 8, 9, 3, \ldots)$ then $Z_1 = 5$, $Z_2 = 8$, $Z_3 = 9$, et cetera. Show that $\{Z_i, i \geq 1\}$ is again a Poisson process with intensity function $\lambda(t)$.
7. This is a construction that has actually come up in my research. Let P be a rate-1 Poisson process on $\mathbb{R}^2$, and let D_i be the distance from the origin to the ith-closest point.

   (a) Prove that $\{D_k^2, k \geq 1\}$ is a Poisson process on $[0, \infty)$ with constant intensity π.

   (b) Show that D_k has density
   $$f(x) = \frac{2\pi^k x^{2k-1} e^{-\pi x^2}}{(k-1)!}$$
   on $[0, \infty)$.
8. Let P be a Poisson point process with constant intensity λ on $\mathbb{R}^d$. For each $p \in P$, let $X_p \in \mathbb{R}^d$ be a random vector. Suppose that the elements of $\{X_p, p \in P\}$ are independent and identically distributed. Show that the set $\{p + X_p, p \in P\}$ is again a Poisson process with constant intensity λ.

   Bonus. What can you say if P does not have constant intensity but the points X_p are still IID? (I have not thought this through in the least.)
9. Grad students join the math department according to a Poisson process of constant intensity λ. They each do coursework, followed by thesis work, and then leave. (Some of them later come back, but that has no bearing on the question.)

Write X_r for the amount of time spent taking courses, and Y_r for the amount of time spent on thesis work, by the r'th student. Suppose that the pairs (X_r, Y_r) are independent for different students.

(a) Find the joint distribution of the numbers $A(t)$ of students still occupied with coursework and $B(t)$ of students working on their theses at time t.

(b) What is the joint distribution if the students instead arrive in pairs, but once they have arrived they behave independently as above ?

10. You are a biology grad student studying the Allis Shad, an endangered fish which spends most of its life in the ocean, but swims upriver to spawn. You have travelled to the Loire valley to try to spot Shad which have been radio tagged. Finding a tagged Shad is quite uncommon, and the job is boring. Luckily, there is excellent local wine in the Loire valley.

    With time t remaining before the end of the day, a fellow grad student offers you a bet based on the arrivals. On seeing a tagged fish, you can say "last fish" if you think it will be the last one to pass before the end of the day. You can only say this once, and you win if you are right ; otherwise you lose. You also lose if no tagged fish pass before the end of the day, or if you don't say "last fish" at all.

    You decide your strategy is to choose the first tagged fish to pass after a specific time s has passed. Assuming that the tagged fish pass according to a Poisson process with constant intensity λ, answer the following questions.

    (a) Find an expression for the probability that you win using this strategy.

    (b) Calculate the value of s which maximizes the probability you win.

    (c) Show that if $\lambda t > 1$, and you choose the best possible value of s, then the probability you win is exactly $1/e$.