Math 547 – Assignment 4

Assigned on March 18, 2018. Due on April 1 at 11 PM, via myCourses.

1 Exercises

Feel free to discuss with others, but write your own solutions in your own words. Explain your work.

1. Suppose P is the transition matrix of an irreducible aperiodic Markov chain with finite state state space V. Prove that P is reversible if and only if for all $n \geq 1$ and all sequences $v_1, v_2, \ldots, v_n$ of states,
$$p_{v_1 v_2} \cdot \ldots \cdot p_{v_{n-1} v_n} p_{v_n v_1} = p_{v_n v_{n-1}} p_{v_{n-1} v_{n-2}} \cdot \ldots \cdot p_{v_2 v_1} p_{v_1 v_n} .$$
2. Let $G = (V, E)$ be a finite graph, and let $B \subset V$. For $v \in V$ write $N(v)$ for the set of neighbours of G, i.e., $N(v) = \{w \in V : \{v, w\} \in E\}$.
Say that a function f is *harmonic with boundary* B if for all $v \in V \backslash B$, $f(v) = \sum_{w \in N(v)} f(w)/N(v)$.
   (a) Show that if f and g are harmonic with boundary B then for all $a, b \in \mathbb{R}$, $af + bg$ is also harmonic with boundary B.
   (b) For $b \in B$ let f_b be the unique harmonic function with boundary B such that $f(b) = 1$ and $f(b') = 0$ for all $b' \in B \setminus \{b\}$. Show that if arbitrary boundary values $(x_b, b \in B)$ are assigned, we can find the unique harmonic function f with these values from the solutions $(f_b, b \in B)$.
3. Find the unique harmonic function with boundary values as shown in the graph in Figure 1. The interior points are a, b, c, d. What is the probabilistic interpretation of this function ?

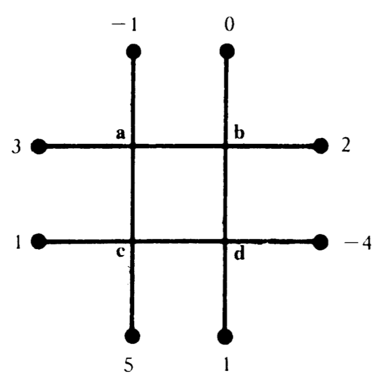

FIGURE 1 – Graph for Question 4.

4. This is a reprise of a question from the midterm. Work out $\mathbb{E}_3[\tau(4)]$ for the below matrix.
$$\begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 0 & 2/3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

5. Show that Polya's theorem for simple random walk (recurrence in $\mathbb{Z}^2$, transience in $\mathbb{Z}^3$) implies that if two random walkers start at the origin and wander independently, then in one and two dimensions they are certain to eventually meet again, but in three dimensions there is positive probability that they won't.

6. Show that Polya's theorem implies that a random walker in three dimensions will eventually hit the line defined by $x = 2$, $z = 0$.

7. Let P be the transition matrix of a Markov chain on the integers $\mathbb{Z}$. In other words, P is the transition matrix of a random walk with state space $\mathbb{Z}$ and with $p_{i,i+1} + p_{i,i} + p_{i,i-1} = 1$ for all i.

   Let $(X_n, n \geq 0)$ and $(Y_n, n \geq 0)$ be independent Markov chains with transition matrix P.

   (a) Prove or disprove that X and Y must eventually meet with probability 1.

   (b) Does the answer change if we assume P is a lazy Markov chain, i.e., $p_{i,i} = 1/2$ for all i?

8. Let $X = (X_t, t \geq 0)$ be continuous-time symmetric simple random walk on $\mathbb{Z}$. In other words, $p_{i,i+1}(h) = h/2 + o(h) = p_{i,i-1}(h)$ as $h \downarrow 0$.

   (a) Show that X is recurrent.

   (b) Assume that $X_0 = 0$. Let T_m be the total time spent at m during an excursion from 0. In other words, writing $\tau = \inf\{t : X_t \neq 0\}$, and $\tau^+ = \inf\{t > \tau : X_t = 0\}$, let T_m be the size (Lebesgue measure) of the set
   $$\{\tau \leq t \leq \tau^+ : X_t = m\}.$$
   Find the distribution of T_m for all m.

9. Suppose X is a transient, continuous-time Markov chain with state space V. Fix $v \in$ and assume that $X_0 = v$. Show that the total time spent in v (i.e. the Lebesgue measure of of $\{t \geq 0 : X_t = v\}$) has an exponential distribution.

10. Let P be the transition matrix of a Markov chain with state space V and let μ and ν be any two distributions on V. Prove that $\|\mu P - \nu P\|_{\mathrm{TV}} \leq \|\mu - \nu\|_{\mathrm{TV}}$. Note that if $\nu = \pi$ is a stationary distribution for P, this shows that advancing the Markov chain can not move it further from stationarity.