Math 547 – Assignment 3

Assigned on February 25, 2018. Due on March 14 at 11 PM, via myCourses.

1 Reading (to do as soon as possible)

1. Section 6.5 (two pages)
2. Review my notes on Section 6.9 and 6.10 (in particular page 14 on convergence to stationarity, which has more detail than was done in class, and page 15 which describes how to build a continuous Markov chain from a discrete one).

2 Exercises

Feel free to discuss with others, but write your own solutions in your own words. Explain your work.

1. Let $G = (V, E)$ be a finite graph with vertices V and edges E. Assume G has no loops (no edges $e = vv$). For $v \in V$ write $\deg(v)$ for the degree of v, i.e., the number of edges incident to v. Show that G is reversible and has stationary distribution $\pi = (\pi_v, v \in V)$ given by $\pi_v = \deg(v)/2|E|$.
2. GS 6.6.6 (the previous question should help you justify your answers) .
3. Compute $p_{11}(t)$ for $P(t) = e^{tG}$ where

$$G = \begin{bmatrix} -2 & 1 & 1 \\ 4 & -4 & 0 \\ 2 & 1 & -3 \end{bmatrix}$$

4. Which of the following matrices can be written as e^G for some generator G ? What does this imply about the discrete time Markov chains with these transition matrices ?

$$\text{(a)} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad \text{(c)} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

5. A pedestrian wishes to cross a single lane of fast-moving traffic. Suppose the number of vehicles that have passed by time t is a Poisson process of rate λ, and suppose it takes time a to walk across the lane. Assuming that the pedestrian can foresee correctly the times at which vehicles will pass by, how long on average does it take to cross over safely. (Hint : consider the time at which the first car passes.)

   How long on average does it take to cross two similar lanes in each of the following cases :

   (a) One must walk straight across, and will not cross if at any time during the crossing, a car would pass on either lane.

   (b) An island in the middle of the road makes it safe to stop half-way.

6. Each bacterium in a colony splits into two identical bacteria after an exponential time of parameter λ, which then split in the same way but independently. Let X_t denote the size of the colony at time t, and suppose $X_0 = 1$. Show that the probability generating function $\phi(t) = \mathbb{E}(z^{X_t})$ satisfies

$$\phi(t) = ze^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} \phi(t-s)^2 ds.$$

Make a change of variables $u = t - s$ in the integral and deduce that $d\phi/dt = \lambda\phi(\phi - 1)$. Hence deduce that, for $q = 1 - e^{-\lambda t}$ and $n = 1, 2, \ldots$,

$$\mathbb{P}(X_t = n) = q^{n-1}(1-q).$$

7. GS 6.5.2
8. GS 6.5.7
9. GS 6.8.1
10. GS 6.8.2
11. GS 6.8.5
12. GS 6.9.3
13. GS 6.6.3 (optional, and no extra credit – only do it if you want a challenge)